Asymptotic Stability of Switched Higher Order Laplacians Implies Sweep Coverage

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Introduction

Recently, several properties in networked sensing and distributed systems have been modeled by various researchers [1–3, 5, 7, 9, 11, 12] using topological spaces and their topological invariants. The unifying theme in these approaches has been that the local properties of a network, as dictated by local interactions among agents, can be captured by certain topological spaces. These spaces are mostly combinatorial in nature and are a generalization of the more familiar graphical models. Moreover, the global properties of the network characteristics correspond to certain topological invariants of these spaces such as genus, homology, homotopy, and the existence of simplicial maps. Examples of such modeling attempts include coverage problems for sensor networks [1–3, 7]; consensus & concurrency modeling in asynchronous distributed systems [9]; and routing in networks without geographical information [5]. One notable characteristic of these studies has been that the topological abstractions preserve many global geometrical properties of the network while abstracting away the redundant geometrical details at small scales. This promises great simplification of algorithms as well as hardware, which is an important requirement for realizing large-scale robust networks.

To elaborate further on this point, consider a network of nodes in a plane, each capable of performing a sensing task within a radially symmetric neighborhood. Then, one can study the problem of *blanket coverage*, i.e. whether the union of the coverage discs about the nodes cover a certain area of interest? This problem has been studied recently using computational homology methods [1–3, 7]. The homological methods permit the study of this problem for sensors which are remarkably minimal, having no means of measuring distance, orientation, or location in their environment.

In this paper, we report our study on dynamic coverage in sensor networks using homological methods by focussing on the mobility of the nodes and the effect of changes in the topology of the underlying network. More precisely, we want to determine if it is possible to provide *sweep coverage* for a given planar domain. By sweep coverage we mean that every point in the environment is required to be covered infinitely often (possibly with some frequency), so that no point is left undetected for long. Thus by deploying a combination of static and mobile sensors, one can produce exploratory patrol-like behavior. This problem is also related to pursuit-evasion problems and to exploration problems in an unknown environment. This problem is well studied and many researchers have proposed different solutions [6]. The aim of our work is not to provide another solution but to provide an online robust *verification* method for any given solution, in order to provide safeguards against node failure and uncertainties in modeling. As explained later, this verification can be implemented with minimal complexity on a large network of homogeneous mobile agents.

Homology and Higher Order Laplacians

The above mentioned criteria for studying coverage are based on the concepts of *simplicial complexes* and their *homology groups* [8]. The simplicial complexes arising in sensor networks are derived from or related to the unit-disk connectivity graphs in networks. In addition to the vertices and edges in graphs, they are made up of higher dimensional objects such as 'triangles' and 'tetrahedrons' that capture tertiary, quaternary or even higher-order relations between basic entities. These objects are collectively known as *simplices*. The homology groups of a simplicial complex X, denoted by $H_k(X)$ for $k \geq 0$, are used to distinguish one simplicial complex from one another by identifying the number of 'holes' of various dimension, contained in them. Each non-trivial member of the homology group in a certain dimension helps identify a corresponding hole in that dimension. Crudely speaking, the dimension of $H_0(X)$ is the number of connected components of X. The dimension of $H_1(X)$ is the number of non-contractible cycles in X, where each cycle encircles a 'puncture' in space. $H_2(X)$ identifies the number of 3-dimensional voids in a space and so on. In sensor networks, the members of the homology groups of certain simplicial complexes help identify *coverage losses* or *network communication holes* [1–3, 7].

Recent advances in computational algebraic topology has enabled the successful verification of these topological results in simulation. However, the algorithms available are not meant for a distributed implementation. Therefore, in order to properly utilize these tools for networked sensing and control, one needs a new approach towards computing topological invariants that is well suited for implementation on real networked systems, with manageable complexity and scalability. In [11, 12], the first steps towards this goal have been taken.

The central objects of this approach are the so-called higher order Laplacian operators on simplicial complexes. It has been shown in [11] that the spectral decomposition of the higher order Laplacian operators is a way to compute homology groups. To understand this operator, we recall that the homology groups are defined as a quotient space $H_k(X) = \ker \partial_k / \mathrm{im}\ \partial_{k+1}$, where ∂_k are the so called *boundary operators*, that linearly map simplices of a particular dimension to simplices at one dimension less. As an example, the familiar incidence matrix in graph theory is the boundary operator ∂_1, mapping edges to vertices. The higher order combinatorial Laplacian $\mathcal{L}_k$ is an operator (matrix) between simplices of the same dimension, and is given by $\mathcal{L}_k := \partial_{k+1}\partial_{k+1}^T + \partial_k^T \partial_k$. Observe that since there are no simplices below dimension zero, $\partial_0 = 0$ and the zeroth order Laplacian is the familiar graph Laplacian $\mathcal{L}_0 = \partial_1 \partial_1^T$. The key observation [4] is that $\ker \mathcal{L}_k \cong H_k(X)$. Again, the readers familiar with algebraic graph theory can verify that the dimension of $\ker \mathcal{L}_0$ is the number of connected components of X, which was earlier identified with $H_0(X)$.

The flow of these Laplacians, roughly described by $\partial\omega/\partial t = -\mathcal{L}_k \omega$, has been used in [11] to detect the absence or presence of network holes. In light of the above discussion and the fact that $\mathcal{L}_k$ is positive definite, it is easy to see that $\omega(t) \to 0$ if and only if $H_k(X) \cong \ker \mathcal{L}_k = \{0\}$, i.e. no holes in dimension k. In order to distinguish between multiple holes, the flow alone does not help unless there is direct method of doing a decentralized spectral decomposition. Such an algorithm has been studied in another related work by the authors [12].

Mobility, Switching Topologies and Dynamic Coverage

Let us now consider the case when the nodes are mobile, giving rise to a switching connectivity graph structure, which induces a switching structure for the simplicial complexes as well. Thus we can study a switched k-Laplacian flow, described by

$$\frac{\partial \omega(t)}{\partial t} = -\mathcal{L}_k^{\sigma} \omega(t),$$

where σ indexes the appropriate simplicial complex whenever there is a change of topology via a *switching signal*. If we are ensured that each simplicial complex encountered during the evolution is hole-free (zero homology), then the switched linear system shows asymptotic stability. However, one can show that the system exhibits asymptotic stability under an even weaker condition, whereby the simplicial complexes encountered in bounded, non-overlapping time intervals are *jointly hole-free*. To understand this condition, let the simplicial complexes encountered in one such interval be $X^1, X^2, \ldots, X^m$. Then, they are said to be jointly hole-free in dimension k, if $H_k(\cup_{i=1}^m X^i) \cong \{0\}$. The proof of this result closely follows the presentation in [10], the details of which have been given in [13].

The consequences of this result for sweep coverage are now easy to explain. The property of being jointly hole-free guarantees that in contiguous bounded time-intervals, the union of the simplicial complexes has zero homology. If the simplicial complexes model coverage properties, where coverage gaps are modeled by holes, then this condition guarantees that the entire region is covered in a bounded time interval. Moreover, the existence of an infinite sequence of contiguous intervals guarantees that each point of the region is visited infinitely often, thus verifying sweep coverage by the mobile network.

We should mention that two issues have not been fully explained in this paper, and we refer the interested reader to a more detailed version of this work [13]. One is that to infer coverage gaps from simplicial complexes, some extra structure is needed. Either, the hole exhibits a certain robustness with respect to the connectivity radius of the underlying graph [2, 3], or some knowledge about the nodes monitoring the boundary has to be assumed [1]. Secondly, we have presented our results without incorporating the effects of coverage losses at the boundaries. To fix this, one needs to compute homology groups for the complexes *relative* to the boundaries nodes. For these reasons, we only have a *verification* algorithm i.e. coverage can be maintained even when the joint-hole free condition is violated. Still, the verification is computationally inexpensive and can be implemented in a decentralized manner. The reason for this is that the k-Laplacians are essentially local averaging or mixing operations, and therefore work in the spirit of gossip algorithms [12].

The switched linear systems studied above are a natural generalization of the work on distributed consensus algorithms [10], based on the standard graph Laplacian from algebraic graph theory. The idea of *joint connectedness* of a set of graphs has been generalized to the idea of when a collection of simplicial complexes are jointly hole-free. Thus, this theme of research allows us to view algebraic graph theory as a special case of the spectral theory of simplicial complexes, which in turn has proven useful in the context of networked sensing and control.

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