Sub-Riemannian Geometry: Basic Ideas and Examples

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Abstract

This tutorial serves as an introduction to the basic ideas in sub-Riemannian geometry. The discussion emphasizes the relevance of this subject from a control theoretic point of view. Some examples of sub-Riemannian geometries such as the Heisenberg geometry and other Carnot groups have also been given.

1 Introduction

Sub-Riemannian geometry is a relatively young area in Mathematics [2]. The geometry is defined on a manifold M, on which every trajectory evolves tangent to a distribution $\mathcal{H}$ of the tangent bundle TM. Riemannian geometry is the special case in which $\mathcal{H} = TM$. Such trajectories are called horizontal curves. A metric, known as the sub-Riemannian metric, is defined as an inner product on the distribution. This metric lets us define the lengths and energies of horizontal curves. The main theme of sub-Riemannian geometry is to study the locally length minimizing horizontal curves or geodesics. The most surprising result of sub-Riemannian geometry is the existence of abnormal geodesics that do not satisfy the normal geodesic equations [2, 3]. In this tutorial we discuss some basic concepts of sub-Riemannian geometries and their importance in the study of control systems and geometric mechanics. This point of view is important for the study of many classical as well as quantum systems. Researchers have studied sub-Riemannian geodesics in Lie groups and have found them useful for the efficient design of quantum logic gates using coherence transfers in nuclear magnetic resonance (NMR) or quantum evolutions in ion trap based quantum computers. [5]. Moreover, sub-Riemannian geometry has a deep connection with optimal control of dynamical systems and Pontryagin's maximum principle [4]. This makes it an important subject and needs to be introduced to researchers outside mainstream mathematics. It is the author's hope to provide here an accessible introduction to this subject. We refer the more curious reader to the splendid text of Montgomery [2] for more details.

2 Geometric Theory of Control Systems

In this section we give an overview of the control problem for systems whose configuration spaces are manifolds. Let M be a manifold that describes the configuration space of a certain dynamical system, also called the *phase* or *state space*. Let TM denote its tangent bundle. A *finite dimensional nonlinear control systems* on a smooth n-dimensional manifold M is a differential equation of the form

$$\dot{x}(t) = f(x, u), \quad x(t_0) = x_0, \tag{1}$$

where $x \in M$, $u(t)$ is a time dependent map from $\mathbb{R}^+$ to a constraint set $\Omega \subset \mathbb{R}^m$, and $f : M \times \mathbb{R}^m \to TM$ is a C^∞ (smooth) or C^ω (analytic) such that for each fixed u, $f(., u)$ is a vector field on M. The map u is assumed to be piecewise smooth or piecewise analytic. Such maps are called *admissible* and the space U of all such maps is called the set of *admissible controls*.

2.1 Controllability and Accessibility

The system in Equation 1 is said to be *controllable* if for any two points x_0 and x_f in M there exists an admissible control $u \in U$ defined on some time interval $[t_0, t_f]$ such that the system with initial condition x_0 reaches the state x_f at time t_f. The notion of controllability is stronger than the notion of *reachability* also known as *accessibility*. The reachable set is roughly defined as the set of points that may be reached by the system by travelling on trajectories from the initial point in a time at most T. By fixing T, the notion of controllability has been weakened as controllability would require the reachable set to be the entire manifold. More precisely, given $x_0 \in M$ and V a neighborhood of x_0, we define $\mathcal{R}^V(x_0, t)$ to be the set of all $x \in M$ for which there exists a control $u \in U$ such that the trajectory of the system belongs to V for all times $t \in [t_0, t]$, with $x(0) = x_0$ and $x(t) = x$. The *reachable set* at time t_f is defined

$$\mathcal{R}^V(x_0, t_f) = \bigcup_{t_0 \leq t \leq t_f} \mathcal{R}^V(x_0, t).$$

A control system is *locally accessible* if given any $x_0 \in M$, the set $\mathcal{R}^V(x_0, t_f)$ contains a nonempty open set of M for all neighborhoods V of x_0 and for every $t_f > 0$. A system is *locally strongly accessible* if given any $x_0 \in M$, then for any neighborhood V of x_0, $\mathcal{R}^V(x_0, t_f)$ contains a nonempty open set of M for any $t_f > 0$ sufficiently small. In this terminology, a system is controllable if given any $x_0 \in M$,

$$\bigcup_{0 \leq t_f < \infty} \mathcal{R}^V(x_0, t_f) = M.$$

2.2 Controllability of Affine Control Systems

For the purpose of studying quantum systems, of special interest are a class of control systems that can be described by differential equations of the form,

$$\dot{x}(t) = X_0(x) + \sum_{i=1}^{m} u_i(t) X_i(x), \tag{2}$$

where each X_i is a vector field on M and the m-tuple $u(u_1, u_2, \ldots, u_m)$ describes an admissible control $u : [t_0, t_f] \to \mathbb{R}^m$. This class of control systems are known as *affine control systems*. The vector field X_0 is called the *drift* term whereas the X_i terms are called the *control* vector fields. We now study the controllability and accessibility conditions for affine control systems. It should be noted that the reachable sets described in the previous paragraphs are hard to compute. Therefore it would be helpful to derive (or know of) a simpler and computable test of controllability or accessibility. It turns out that for affine control systems of the form in Equation (2), such a test does exist [**?**]. The underlying method makes use of the Lie algebra generated by the set of vector fields $\{X_0, X_1, \ldots, X_m\}$.

We now recall some basic definitions from the theory of Lie algebras. For more details see [**?**]. A *Lie algebra* is a real vector space V with a multiplication operation $[\cdot, \cdot] : V \times V \to V$ that satisfies the following properties.

1. *Skew Commutativity:* $[A, B] = -[B, A]$ for any $A, B \in V$.
2. *Bilinearity:* If $A, B, C \in V$ and $c \in \mathbb{R}$ then

$$\begin{aligned} [A, B + C] &= [A, B] + [A, C] \\ .\ [A + B, C] &= [A, C] + [B, C] \\ c[A, B] &= [cA, B] = [A, cB] \end{aligned}$$

3. *Jacobi Identity:* $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$ for all $A, B, C \in V$.

In the case of vector fields, the multiplication operation is the familiar *Lie bracket* given by

$$[X, Y] = YX - XY.$$

In more familiar terms, let the two vector fields be expressed in local coordinates as

$$X(x) = \sum_{i=1}^{n} a_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i}, \quad Y(x) = \sum_{i=1}^{n} b_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i},$$

then their Lie bracket is given by

$$[X, Y](x) = \sum_{i=1}^{n} c_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i}, \text{where } c_i = \sum_{j=1}^{n} \frac{\partial a_i}{\partial x_j} b_j - \frac{\partial b_i}{\partial x_j} a_j.$$

In the case of linear vector fields, the Lie bracket is simply the matrix commutation of the two matrices that define the vector fields. Let $X = Ax$ and $Y = Bx$ where $A, B \in \mathbb{R}^{n\times n}$ then

$$[X, Y](x) = (AB - BA)x.$$

Now, let $\mathcal{L}$ be the smallest Lie algebra of vector fields on M that contains $X_0, X_1, \ldots X_m$, *i.e.*, the set of vector fields generated by all iterated Lie brackets of the form $X_0, X_1, \ldots, X_m$, $[X_0, X_1], [X_1, X_2], \ldots, [X_0, [X_1, X_2]], [X_0, [X_2, X_3]], \ldots,$ and so on. This set is also referred to as their *Lie Hull*. When these vector fields define an affine control system, $\mathcal{L}$ is also called the *accessibility Lie algebra* of the system. At each point $x \in M$ we define a *distribution* or a plane in the tangent space as

$$L(x) = \{\mathrm{span}\, Z(x) : Z \in \mathcal{L}\} \subseteq T_xM.$$

Similarly, let $\mathcal{L}_0$ be the smallest Lie algebra of vector field on M that contain $X_1, \ldots, X_m$ (excluding the drift term X_0) and that satisfies the additional condition that $[X_0, Z] \in \mathcal{L}_0$ for all $Z \in \mathcal{L}_0$. Let

$$L_0(x) = \{\mathrm{span}\, Z(x) : Z \in \mathcal{L}_0\} \subseteq L(x).$$

We now summarize the most important result for affine control systems [**?**].

Theorem 2.1 *For the affine control system described by Equation 2 we have the following conditions.*

1. **Local accessibility.** *The system has local accessibility if and only if* $\dim L(x) = \dim(T_xM) = n$ *for all* $x \in M$.
2. **Local strong accessibility.** *The system has local strong accessibility if and only if* $\dim L_0(x) = n$ *for all* $x \in M$.
3. **Controllability (Chow's Theorem).** *If the system is locally accessible and drift-less, i.e.* $X_0 = 0$ *then the system is controllable.*

These spanning conditions are commonly known as the Lie Algebra Rank Conditions (LARC). For the special yet important case of linear systems the weaker notion of accessibility *does* imply controllability as demonstrated below. Let the system be given by

$$\dot{x} = Ax + \sum_{i=1}^{m} b_i u_i,$$

where $A \in \mathbb{R}^{n\times n}$ and $b_i \in \mathbb{R}^n$. The tangent space $T_x\mathbb{R}^n \simeq \mathbb{R}^n$. Let us compute the Lie algebras $\mathcal{L}$ and $\mathcal{L}_0$ for this system. Here Ax is the drift term whereas the constant vectors b_i are the control vector fields. In the course of computation, we ignore the iterated brackets that generate 0 or those that lie in the span of previously computed brackets. The Lie bracket $[b_i, Ax] = Ab_i$, while $[b_i, b_j] = 0$. Continuing the Lie bracket computation we have $[[b_i, Ax], Ax] = [Ab_i, Ax] = A^2 b_i$, $[[[b_i, Ax], Ax], Ax] = A^3 b_i$ and so on. By the Cayley-Hamilton theorem each A^n is linearly dependent on $A^0, A^1, \ldots, A^{n-1}$, therefore nothing new is produced in iterated brackets after obtaining $A^{n-1}b_i$. It is easy to see that at each $x \in \mathbb{R}^n$

$$L_0(x) = L(x) = \text{span}\{b_1, b_2, \ldots, b_m, Ab_1, Ab_2, \ldots Ab_m, \ldots A^{n-1}b_1, A^{n-1}b_2 \ldots, A^{n-1}b_m\}.$$

If we stack the column vectors b_i's in a matrix $B = [b_1, b_2, \ldots, b_m] \in \mathbb{R}^{n \times m}$, then the spanning test of the accessibility Lie algebra is equivalent to the classical controllability condition on the rank of the matrix

$$\mathcal{C} = [B, AB, A^2B, \ldots, A^{n-1}B].$$

We say that the linear system

$$\dot{x} = Ax + \sum_{i=1}^{m} b_i u_i = Ax + Bu,$$

is controllable if $\text{rank}(\mathcal{C}) = n$. It is clear that the method of studying controllability or accessibility by Lie algebras is much simpler than direct computation of reachable sets. We will make use of this method at several occasions, especially for verifying the existence of geodesics in certain sub-Riemannian geometries.

3 Main Ideas in Sub-Riemannian Geometry

In this section, we give a short introduction to the area of sub-Riemannian geometry. We follow closely the treatment given in [2], which can be referred for a comprehensive introduction to this subject. Informally, a sub-Riemannian geometry is a type of geometry in which the trajectories evolve tangent to a horizontal plane inside the tangent plane only. The main theme of this subject is the study of geodesics arising in such a geometry. In particular, the existence of minimizing geodesics that do not satisfy the so-called normal geodesic equations is an outstanding result of this subject. Such singular or abnormal geodesics are not found in the more familiar Riemannian geometry. The existence of such geodesics is of extreme importance to control theory, where such investigations have far reaching applications in many areas such as robotics, geometric mechanics and quantum control. From a purely mathematical point of view, sub-Riemannian geometry is a more generic way of looking at Riemannian geometry, which arises as a special case in the sub-Riemannian setting. Together, this theoretical generalization and its various applications make sub-Riemannian geometry an important field of study. It is also referred to as the Carnot-Caratheodory geometry in the literature.

3.1 Metrics and Lengths

A sub-Riemannian geometry is fully described by the following triplet.

1. Manifold: A manifold Q of dimension n, with the usual smooth or differentiable structure on it. In particular T_pQ denotes the tangent space at a point $p \in Q$ and $TQ \cup_{p\in M} T_pQ$ is the tangent bundle on Q.

2. Distribution: A distribution $\mathcal{H} \subseteq TQ$, defined as a linear sub-bundle of TQ. In other workds, at each point $p \in Q$, a k-plane $\mathcal{H}_p$ of dimension $k \leq n$ spans a linear subspace of T_pQ. The collection of all such k-planes is the distribution $\mathcal{H}$. The distribution is also called the horizontal space.

3. Metric: A metric or an inner product $< v, w >_p$ on the distribution between any two vectors $v, w \in \mathcal{H}_p$.

The usual Riemannian geometry is a special case in this setting when $\mathcal{H} = TM$ and the metric corresponds to the Riemannian metric on the manifold. It is in this sense that sub-Riemannian geometry is a generalization of the standard Riemannian geometry. Let $\gamma : [0, T] \to Q$ be a horizontal curve. The length of this curve is given by

$$l(\gamma) = \int_0^T \|\dot{\gamma}(t)\|^2 dt,$$

where $\|\dot{\gamma}\| < \dot{\gamma}(t), \dot{\gamma}(t) >^{1/2}$. The distance between two points $q_1, q_2 \in Q$ is given by

$$d(q_1, q_2) := \inf_{\gamma} l(\gamma)$$

where the inf is taken over all absolutely continuous curves joining q_1 to q_2. Note that this is a larger class of curves than that of all smooth curves. The curve γ^* corresponding to the inf is called the minimizing geodesic. Since, our aim is to study the geodesics, i.e. locally length minimizing horizontal curves in this geometry, we take a closer look into what it means to minimize the length (locally). First note, that the energy associated with the motion on a curve γ is given by

$$E(\gamma) = \int_0^T \frac{1}{2} \|\dot{\gamma}\|^2.$$

If we minimize this energy, the length is also minimized. To see this we present the standard argument from Riemannian geometry [**?**] as follows. Write the length as,

$$l(\gamma) = \int 1.\|\dot{\gamma}\| dt \leq \sqrt{T} \sqrt{\int \|\dot{\gamma}\|^2 dt} = \sqrt{2E(\gamma)T},$$

which is obtained by the Cauchy-Schwartz inequality. It is also worth noting that equality is achieved for constant speed curves only (again by the Cauchy-Schwartz inequality).

It is also important to understand how a metric is properly defined in the sub-Riemannian geometry. In the usual Riemannian geometry, the Riemannian metric is defined by a contra-variant 2-tensor, i.e. a section of the bundle $S^2(T^*Q)$, where T^*Q denotes the cotangent bundle of the manifold Q and $S^2(T^*Q)$ is the symmetrization of $T^*Q \otimes T^*Q$. This is not applicable in sub-Riemannian geometry because the metric defined above is defined for any two co-vectors in the cotangent space, whereas in sub-Riemannian geometry the co-vectors are confined to a distribution $\mathcal{H} \subset TQ$. We therefore use the metric as follows.

Definition 3.1 *A* **sub-Riemannian metric** *is defined as a contra-variant symmetric* 2*-tensor, i.e. a section of the bundle* $S^2(TQ)$*, whose rank* $k < \dim(Q)$ *is the rank of the distribution.*

Note that because of the rank deficiency, this cannot Hence cannot be inverted to get the Riemannian metric described above. This metric is also called a *cometric*. Let us define a bilinear form $\ll \cdot,\cdot \gg: T^*Q \otimes T^*Q \ \to \mathbb{R}$, as an inner product on co-vectors. This bilinear form serves as our cometric and behaves in the following way. The cometric induces a symmetric map $\beta = \beta^* : T^*Q \to TQT^{**}Q$ by $w_1(\beta(w_2)) = \ll w_1, w_2 \gg$ for $w_1, w_2 \in T^*Q$. This map β lets us go back and forth between TQ and T^*Q.

To see where this cometric comes from, let us consider a particular sub-Riemannian geometry, defined using a manifold Q, $\mathcal{H} \subset TQ$ and the inner product on $\mathcal{H}$, $< \cdot,\cdot >$. The map β is *uniquely* determined as follows. First note that the the image of the cometric must be the entire distribution, i.e. $Im(\beta_q) = \mathcal{H}_q$ for $q \in Q$. Therefore, the inner product $< \cdot,\cdot >$ on $\mathcal{H}$ and β are related by $p(v) = < \beta_q(p), v >$, where $v \in \mathcal{H}$ and $p \in T_q^*Q$. This gives us a unique description for the cometric. Conversely, it can be shown that any cometric gives a sub-Riemannian geometry $\mathcal{H}$ and $< \cdot,\cdot >$.

It is important to realize that the sub-Riemannian metric for a particular distribution on the manifold can also be described as the metric associated with a control system of the form given in Equation 2, with $X_0 = 0$. Let $\mathcal{H}$ be the distribution locally spanned by $\{X_1, X_2, \ldots, X_m\}$. At a point $x \in Q$ and any $Z \in T_xQ$, we set

$$g_x(Z) = \inf\{u_1^2 + \ldots + u_m^2 : u_1X_1 + \ldots + u_mX_m = Z\}.$$

then g_x is a positive definite quadratic form defined locally on the subspace $\mathcal{H}_x = \mathrm{span}\{X_1(x), \ldots, X_m(x)\} \subseteq T_xQ$. If $Z \notin \mathcal{H}_x$ then we set $g_x(Z) = \infty$. To see why this is a proper definition, consider the map $\sigma_x : \mathbb{R}^m \to T_xQ$ given by $(u_i, \ldots, u_m) \mapsto u_1X_1(x) + \ldots + u_mX_m(x)$. Then the restriction of σ_x to $\ker \sigma_x^{\perp}$ is a linear isomorphism onto $\mathcal{H}_x$. Now construct an inverse mapping $\rho_x : H_x \to \ker \sigma_x^{\perp}$ by

$$g_x(Z) = \begin{cases} \|\rho_x(Z)\|^2, & \text{if } Z \in \mathcal{H}_x; \\ +\infty, & \text{otherwise.} \end{cases}$$

By this construction, we say that g is the sub-Riemannian metric associated with the affine control system

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t)X_i(x),\ x \in Q,$$

We now have an understanding of metrics and lengths on sub-Riemannian manifolds and can study the geodesics as length minimizing horizontal curves.

3.2 Normal Geodesics and Hamiltonian dynamics

Questions about the existence of geodesics between two arbitrary points of a manifold in a sub-Riemannian geometry can be answered by using the Lie algebraic methods described in Section 2. Once we get a controllability certificate, we can look for geodesics in the geometry. In the following paragraphs, we explore role of Hamiltonian methods in obtaining these geodesics.

Let us define the Hamiltonian as

$$H(q,p) = \frac{1}{2} \ll p, p \gg,$$

where $q \in Q$ and $p \in T_q^*Q$. This is also known as the sub-Riemannian Hamiltonian or the kinetic energy of the system. For a horizontal curve $\gamma(t)$, and for some $p \in T_{\gamma(t)}^*Q$, let $\dot\gamma(t) = \beta_{\gamma(t)}(p)$. Then

$$H(q,p) = \frac{1}{2}\|\dot\gamma(t)\|^2.$$

In this way the β uniquely determines the Hamiltonian and the Hamiltonian uniquely determines the sub-Riemannian structure by polarization. To compute the Hamiltonian, we set up a local frame $\{X_a\}, a = 1 \ldots k$ of vector fields on $\mathcal{H}$. For each $X \in \mathcal{H}$ define a momentum function $P_X : T^*Q \to \mathbb{R}$ by $P_X(q,p) = p(X(q))$. In local coordinates let $X_a = \sum X_a^i \frac{\partial}{\partial x^i}$, and its momentum function be given by $P_a(x,p) = \sum X_a^i(x)p_i$ where $p_i = P_{\partial/\partial x^i}$. Notice that (x^i, p_i) give canonical coordinates on the cotangent bundle T^*Q.

Let $g_{ab}(q) =< X_a(q), X_b(q) >_q$ be a matric defined by the inner product on $\mathcal{H}$. Let $g^{ab}(q)$ be the $k \times k$ matrix inverse of $g_{ab}(q)$. It is easy to see that

$$H(q,p) = \frac{1}{2}\sum g^{ab}(q)P_a(q,p)P_b(q,p).$$

If $\{X_a\}$ make an orthonormal frame on $\mathcal{H}$, relative to the sub-Riemannian metric, then $H(p,q) = \frac{1}{2}\sum P_a^2$. We can now employ the standard techniques from Hamiltonian dynamics to get, what are known as the *normal geodesic equations.*

$$\dot x^i = \frac{\partial H}{\partial p_i}, \quad \dot p_i = -\frac{\partial H}{\partial x^i}.$$

We now give the normal geodesics theorem.

Theorem 3.1 (Normal geodesics theorem) *Let $(\gamma(t), p(t))$ be a solution to the Hamilton's equations on T^*Q and let $\gamma(t)$ be its projection to Q. Then every sufficiently short arc of γ is a minimizing sub-Riemannian geodesic. Moreover, γ is the unique minimizing geodesic joining the endpoints.*

As one can anticipate from the term *normal*, there may exist *abnormal* or *singular* geodesics in certain sub-Riemannian geometries that do not satisfy the normal geodesics equations. We will describe these abnormal geodesics later. First let us try to give a heuristic proof of the above theorem. Let us pick a Riemannian metric g that is compatible with the sub-Riemannian metric $< .,. >$ on $\mathcal{H}$. Now, split $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$, where $\mathcal{V}$ is orthogonal to $\mathcal{H}$ with respect to g. Now form a family of Riemannian metrics

$$g_\lambda = g_{\mathcal{H}} \oplus \lambda^2 g_{\mathcal{V}}, \quad \lambda \to \infty,$$

all compatible with the Riemannian metric g. Form an orthonormal frame $\{X_a\}$ for $\mathcal{H}$ and $\{(1/\lambda)X_i\}$ for $\mathcal{V}$. We can then obtain a family of Hamiltonians

$$H_\lambda = \frac{1}{2}(\sum P_a^2 + \frac{1}{\lambda^2}\sum P_i^2).$$

Now by letting $\lambda \to \infty$, one should expect that H_λ tends to a Hamiltonian H governing the normal geodesic equations. This proof however does not work for the case of geometries where abnormal geodesics may also be present. The actual proof of the theorem has been omitted for the sake of brevity, and has been given in detail in [2].

4 Examples

In this section we give several examples of sub-Riemannian geometries and study their properties using the basic ideas developed above.

4.1 Heisenberg Geometry

To elaborate the abstract setting of sub-Riemannian geometry in this section, let us analyze in the Heisenberg Geometry. Here $Q = \mathbb{R}^3$. Consider the 1-form on $\mathbb{R}^3$ given by

$$\theta(x, y, z) = dz - \frac{1}{2}(xdy - ydx),$$

The distribution corresponding to this annihilating form is $\mathcal{H}_{(x,y,z)} = \{\theta(x, y, z) = 0\}$. For any two vectors $v, w \in \mathcal{H}_(x, y, z)$ we define the Sub-Riemannian metric on $\mathcal{H}$ as $< v, w >= v_1 w_1 + v_2 w_2$. In other words

$$ds = \sqrt{dx^2 + dy^2},$$

and hence movement in the z direction does not contribute towards the length of a geodesic. Now construct an orthonormal frame for $\mathcal{H}$ at (x, y, z) is given by

$$X = \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z},$$

Now compute the Lie bracket. We find that

$$[X, Y] = Z = \frac{\partial}{\partial z}, \quad [X, Z] = [Y, Z] = 0.$$

It is easy to see that X, Y and $[X, Y]$ span the whole tangent bundle. Therefore by Theorem 2.1 any two points in $\mathbb{R}^3$ can be connected by a horizontal curve in the Heisenberg geometry. Now compute the momentum functions

$$P_X = p_x - \frac{1}{2} y p_z, \quad P_Y = p_y + \frac{1}{2} x p_z,$$

where (x, y, z, p_x, p_y, p_z) give coordinates on $T^*Q = \mathbb{R}^3 \oplus \mathbb{R}^3$. The Hamiltonian is given by $H = \frac{1}{2}(P_X^2 + P_Y^2)$. From this we Hamilton's equations.

$$\begin{aligned}
\dot{x} &= P_X, \\
\dot{P}_X &= -P_Z P_Y, \\
\dot{y} &= P_Y, \\
\dot{P}_Y &= P_Z P_X, \\
\dot{z} &= -\frac{1}{2} y P_X + \frac{1}{2} x P_Y, \\
\dot{P}_Z &= 0.
\end{aligned}$$

So x, y, P_X, P_Y evolve independently of z, and with a constant P_Z. Integrating the rest of the equations give the geodesics. Some geodesics are depicted in Figure **??**. A detailed description of the Heisenberg geodesics is given in [**?**, **?**]. Without much detail, it is worth noting that the Hiesenberg sphere, i.e. all points at unit sub-Riemannian length is shaped like an apple. Also note that the sphere is singular at z-axis. All points along z-axis are conjugate points to the origin. As mentioned above, the vector fields X, Y and $Z = [X, Y]$ make a basis for the tangent bundle. In fact, they generate a Lie algebra called the Heisenberg algebra H_3. Since $[X, Z] = 0, [Y, Z] = 0, [[X, Y], Y]] = 0$ and so on, we have a *nilpotent Lie algebra* of depth 2. The exponential map provides a diffeomorphism between the nilpotent Lie algebra and its unique simply connected Lie group. At identity,

$$\gamma_v(t) = \exp(t(v_1, v_2, v_3)) = (tv_1, tv_2, tv_3)$$

Therefore Heisenberg algebra gives the Heisenberg group with the group law

$$(x_1, y_1, z_1).(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1))$$

Also, $\frac{d}{dt}|_{t=0}(q.\gamma_v(t)) = v_1 X(q) + v_2 Y(q) + v_3 Z(q)$. So X, Y, Z make a basis for the space of left-invariant vector fields on the group. Finally, $\mathcal{H} = \text{span}\{X, Y\}$ is also left-invariant with respect to the group law.

4.2 Carnot Groups

These observations are not specific to the Heisenberg group only. Let G be a Lie group and $\mathcal{L}$ be its Lie algebra, i.e. the space of left invariant vector fields on the group. Take $V \subset \mathcal{L}$ a left-invariant distribution. If V Lie generates $\mathcal{L}$, then we have the same condition as the bracket-generating condition of Chow's Theorem. The restriction to V of an inner product on $\mathcal{L}$ gives a sub-Riemannian metric. The action on G by left translations is an action by isometries. Examples of such Lie groups are the so called Carnot groups. We have

$$\mathcal{L} = V \oplus V_2 \oplus \ldots \oplus V_r,$$

where $V_{i+j} := [V_i, V_j]$, we say that V Lie generates $\mathcal{L}$. The Heisenberg group is the simplest Carnot group of step 2.

5 Abnormal geodesics

We close this section by saying a few words about abnormal geodesics in sub-Riemannian geometry. We give an explicit example of such a geometry. We consider the *Martinet distribution* on $\mathbb{R}^3$ given by

$$\Theta(x, y, z) = dz - \frac{1}{2}y^2 dx,$$

Each horizontal plane is spanned by vector fields

$$X = \frac{\partial}{\partial x} - \frac{1}{2}y^2\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}$$

$[X, Y] = -y\frac{\partial}{\partial z}$, $[[X, Y], Y] = -\frac{\partial}{\partial z}$. Let the *Martinet surface* be given by $\mathbf{\Sigma} = \{y = 0\}$. Off this horizontal plane, the $X, Y, [X, Y]$ span $T\mathbb{R}^3$ while $X, Y, [[X, Y], Y]$ span $T\mathbb{R}^3$ on $\mathbf{\Sigma}$. Again, by Chow's Theorem, any two points in this geometry are horizontally connected. One can show that on the x-axis, any horizontal curve $\gamma(t) = (t, 0, 0)$ on $\mathbf{\Sigma}$ is abnormal. To see that, pick vector fields X, Y such that X is tangent to Martinet curves, and Y is transverse to $\mathbf{\Sigma}$. Let

$$X = (1 + y\psi_1)\frac{\partial}{\partial x} + y\psi_2\frac{\partial}{\partial y} + y\psi_3\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}.$$

It can be proven that if the x-axis is the projection of a solution to Hamilton's equations for the sub-Riemannian Hamiltonian, then $\psi_1 = 0$ along x-axis. It turns out that this is a contradiction.

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