Spectral Properties of Expansive Configuration Spaces: An Empirical Study

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Abstract—Motion planning in high dimensional configuration space is an intractable problem. Sampling based motion planning is considered as one of the best ways to tackle the “curse of dimensionality” and to model the configuration space. In these methods (e.g. probabilistic roadmap (PRM)), a graph is produced by random nodes of valid configurations. As the number of the nodes \( N \) increases, for expansive configuration spaces the failure probability of PRM planner exponentially approaches zero; but computing expansiveness property is not feasible for most interesting problems. Another major hurdle is narrow passages detection. Therefore, we need easily computable ways to characterize the properties of configuration spaces. We propose a new framework for narrow passage detection using spectral analysis of the graph Laplacian of the PRM. We give empirical evidences to show that eigenvalues and eigenvectors reveal useful information about number and size of narrow passages, visibility and expansiveness. Simulations on various motion planning scenarios are done to verify the framework.

I. INTRODUCTION

Sampling based motion planning is an extremely effective and popular motion planning technique in robotics [1], [2], [3], [15], [17]. If complete knowledge of configuration space is available, it is possible to find optimal set of samples to cover the whole free configuration space. But obtaining the complete knowledge of a configuration space is not an easy task, specially when we go to higher dimensions. In sampling based motion planning, a road map infers and represents the approximate model of the configuration space. This technique is specially useful where navigation by analytical methods is hard due to the complexity of the configuration space or due to the unavailability of its exact representations [4], [5], [7], [9]. The complexity of a sampling based motion planner depends on its sampling strategy. The closer to optimal is the sampling strategy, the better is the planner. Till now, there is no single optimal motion planner in existence. Performance and completeness of the motion planner is dependent on configuration space properties.

An important configuration space property is the \((\alpha, \beta, \varepsilon)\)-expansiveness. This expansiveness captures the difficulty of motion planning and the requirements on sampling of the configuration space. Parameters \( \alpha, \beta \) and \( \varepsilon \) dictate the extent to which the visibility region of the sampled configurations can be expanded (see formal definition in the next section). For a configuration space, characterized by high values of \( \alpha \) and \( \beta \), sampling easily captures the connectivity of the space by a roadmap small size.

The computational complexity of the motion planner is directly related to the complexities associated with the geometry of the configuration space. In addition to ambient dimensions and degrees of freedom, one limiting geometrical feature is the presence of narrow passages. Configuration spaces that contain narrow passages exhibit poor expansiveness, characterized by lower values of \( \alpha, \beta, \) and \( \varepsilon \). Performance of the motion planner suffers dramatically in poorly expansive environments and therefore requires a high sampling density of milestones in narrow regions. In [7], it is proved that as the number of configuration samples \( N \) increases; the failure probability of motion planner exponentially approaches zero. These parameter are very important for characterizing the configuration space. Unfortunately, computing these parameters is not feasible for most interesting problems and there are also non-expansive spaces in motion planning applications like manipulation and legged locomotion [6]. Therefore, there is an inevitable need for easily computable ways to characterize the properties of configuration spaces.

In sampling based motion planning, narrow passage detection is a major hurdle. After detecting narrow passages, sampling distribution is found. Again narrow passages are an important feature to be classified and located for motion planning. Algorithms like the Bridge Test, Small Step Retraction Method and Free Space Dilation address this issue in sampling based motion planning [8], [10], [16].

There are many fields and applications in which spectral analysis has been used for studying spaces. These include manifold learning, image segmentation and spectral clustering [11], [12], [13], [14]. These techniques have very close relationship to spectral graph theory. In this paper, we propose a new framework for analyzing configuration space properties and narrow passage detection using spectral graph theoretic techniques. We present an alternative viewpoint of looking at motion planning difficulties. Our analysis is based on tools of harmonic analysis and graph Laplacians. As a dual to expansiveness in the spatial domain, we present a frequency domain approach to understanding these diffi-
Fig. 1. LOOKOUT(G) and visibility of configurations to F\G.

cultures, in particular narrow passages and other geometrical and topological complexities. The higher order topological complexities will appear in a future work although this connection is immediately clear from a spectral understanding of topology using Hodge theory [24], [23], [26].

II. EXPANSIVE CONFIGURATION SPACES & PRMs

Notion of expansiveness [7] is used to characterize a configuration space in roadmap based motion planning techniques. A better expansive space can easily expand to new visibility regions by sampling new points. Narrow passages make a configuration space less expansive, so that the expansion of new regions is not possible and random sampling fails to cover the whole free space. There are three parameter $\alpha, \beta$, and $\epsilon$ used in the definition of expansiveness. Free space $F$ is $\epsilon$-good, if every configuration $q$ sees at least $\epsilon \mu(F)$, where $\mu(F)$ denotes volume of the free space. The free space $F$ is $(\alpha, \beta, \epsilon)$-expansive if each of its connected components $F' \subset F$ satisfies $\epsilon$-goodness condition and for any subset $G \subset F'$, the set

$$\text{LOOKOUT}(G) = \{ q \in G \mid \mu(V(q)\setminus G) \geq \beta \mu(F'\setminus G) \}$$

has volume $\mu(\text{LOOKOUT}(G)) \geq \alpha \mu(G)$. Refer to Figure 1 for a geometric depiction of the sets used in this definition.

PRM planning algorithm works in two distinct phases. In the off-line learning phase, a PRM is generated by randomly sampling the configuration space and then looking for collision-free configurations called milestones [18], [19]. The valid configurations are then tested in pairs to find those configurations that are neighbors in connectivity via a collision free trajectory. These milestones and their mutual reachability information make up a graph made of milestone nodes and path edges. This creates the so-called roadmap and captures the connectivity of free space.

In the online query phase, a source and goal configuration pair is tested against the PRM and the roadmap is searched for the shortest path connecting the two query configurations by a graph search algorithm. For a PRM to be probabilistically complete, the free space must satisfy visibility properties of expansiveness. This expansiveness captures the difficulty of motion planning and the requirements on sampling of the configuration space. Referring to [7], [9], [20], the probability of failure of the PRM planner has an upper bound given by

$$Pr_e < \frac{c_1}{\epsilon \alpha} \exp \left( \frac{c_2 \epsilon \alpha}{\beta} \right)$$

where $c_1, c_2$ and $c_3$ are positive constants and $N$ is the number of sampled milestones. Roughly, this upper-bound states that as the number of sampled nodes $N$ increases, the failure probability of PRM planner exponentially approaches zero.

III. GRAPH LAPLACIANS AND THEIR SPECTRA

A PRM graph $G$, can be parameterized by an ordered pair $(V, E)$ of vertices and edges, where $V$ refers to the sampled milestones (also referred to as nodes) and $E$ refers to the set of collision-free paths that exists between vertices. $G$ can also be represented by an adjacency matrix $A$, defined by

$$A_{ij} = \begin{cases} 0, & i = j; \\ 1, & i \neq j \text{ and } v_i \sim v_j; \\ 0, & \text{otherwise}. \end{cases}$$

A another useful representation of a graph is the Laplacian matrix. The Laplacian matrix is defined by $L = D - A$, where the degree matrix $D$ is a diagonal matrix capturing the number of vertices adjacent to each vertex. Therefore, entry-wise the matrix is given by

$$L_{ij} = \begin{cases} \deg(v_i), & i = j; \\ -1, & i \neq j \text{ and } v_i \sim v_j; \\ 0, & \text{otherwise}. \end{cases}$$

Laplacian matrix reveals insightful information about the graph. Let $N$ be the total nodes in a graph. The spectral decomposition of the Laplacian matrix

$$L \psi_i = \lambda_i \psi_i, \ i = 0, \ldots, N - 1,$$

gives $N$ number of eigenvalues $\lambda_i$ and corresponding eigenvectors $\psi_i$. Since $L$ is always positive semi-definite, $\lambda_i$ are real and non-negative. We order the eigenvalues as $\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{N-1}$. In particular $\lambda_0$ is always zero and corresponding eigenvector is the constant vector $1 = [1, 1, \ldots, 1]^T$. The number of connected components of $G$ is given by the number of eigenvalues equal to zero. The smallest non-zero eigenvalue gives the algebraic connectivity of the graph. Algebraic connectivity reflects how well-connected the graph is.

Note that the eigenvalues are concave functions of $L$. In particular, $\lambda_1$ is a concave function of $L$ in the space orthogonal to constant vectors. It is given by

$$\lambda_1(L) = \inf_{\omega \in \mathbb{C}} \frac{\omega^T L \omega}{\omega^T \omega}.$$ 

Similar definitions can be given for other eigenvalues by choosing the minimization orthogonal to the space spanned by other eigenvectors of lower eigenvalue.

Incidentally, this graph Laplacian is a direct discretization of the Laplacian operator from analysis [21], [22], [23], [24]:

$$\Delta f = \text{div}(\text{grad } f).$$
The eigenvalue problem for this analytical Laplacian on a domain $M$ is to find all numbers $\lambda$, for which there exist a nontrivial solution $\phi \in C^2(M)$ (harmonics) to

$$\Delta \phi = -\lambda \phi.$$ 

Variants of this problem are known as the Neumann and Dirichlet eigenvalue problems [25]. Note the similarity with Equation (2).

One way to look at this connection is the so-called node level expression of the Laplacian applied to the vertices,

$$L(v_i) = \sum_{v_i \sim v_j} (v_i - v_k).$$ 

This is a local averaging formula and works in the same spirit as the heat kernel $e^{-\Delta t}$ on a continuous domain. By using this analogy to heat propagation in a medium, one can visualize how the discrete Laplacian settles to particular harmonics.

When the graph is a probabilistic roadmap, this information becomes even more revealing. In the next section, we do spectral analysis of some examples and demonstrate the relationship between the roadmap spectra and the geometry of the underlying configuration space. This then relates to the discovery of the number and geometric dimensions of narrow passages in the configuration space.

IV. Unraveling the Configuration Space Using Graph Spectrum

We begin by studying two very simple examples, the first example is shown in Figure 2. The robot lives in a 2D space made up of two chambers connected by a narrow passage. It is also free to rotate about a pivot at its center. Thus the free configurations make up a bounded subset of $\mathbb{R}^2 \times S^1$ [1], [2], [3]. The second example is a 3D motion planning problem as depicted in Figure 2. The 3D ambient space is obstructed by a wall that has an opening. The size of the opening creates motion planning difficulties for a small robot that has a nontrivial physical shape.

We construct the roadmap for these problems according to the standard algorithm [1], and setup the milestones and edges. We assume that the samples adequately cover the space and provide a good approximation of its structure, this is guaranteed by choosing an acceptably low failure probability according to inequality (1). We next setup the graph Laplacian and compute its eigenvector decomposition. Note that in all the following figures, sampled nodes have been colored as red or blue to indicate that the sign of the corresponding attribute of interest is positive or negative respectively. Also, the intensity of a color is proportional to the magnitude of that attribute. The blacker a node, the closer that attribute is to zero for that node, for example this visualization technique will be used to study components of a eigenvector.

A. Spectral Gap and Narrow Passages

In Figure 3, we distort a perfect square (i.e. no obstacles) progressively by introducing a neck between two chambers.

Fig. 2. Motion planning examples. [Right] A 2D motion planning example. [Left] A 3D motion planning example with a narrow passage.

Fig. 3. A series of 2D motion planning examples showing spectral gap $\lambda_1$. Dots represent valid configurations that are color modulated by the intensity of the second Laplacian eigenvector $\psi_1$.

Fig. 4. A series of 3D motion planning examples showing spectral gap $\lambda_1$. Dots represent valid configurations that are color modulated by the intensity of the corresponding Laplacian eigenvectors $\psi_1$. [Top left] One obstruction ($\psi_2$). [Top right] Two obstruction ($\psi_2$). [Bottom left] Three obstruction ($\psi_3$). [Bottom right] Four obstruction ($\psi_4$).
RESULTS verify that spectral gap; eigenvalue sequences and their difference (approximate derivative) are given in Table II and Figure 5 respectively.

We keep distorting the geometry until the space breaks into two. The total volume is preserved during this distortion.

By looking at the eigenvalues of the Laplacian, we can predict the number of obstruction (narrow passages) from the spectral gap. Roughly, spectral gap $\Lambda$ is the biggest jump in consecutive eigenvalues of the spectral sequence. Spectral gap $\Lambda = \lambda_{i+1} - \lambda_i$ indicates $i$ narrow passages in a connected space. If there is one narrow passage i.e. $\Lambda = \lambda_2 - \lambda_1$, then $\lambda_1$ is proportional to the width of the passage. In graph theory, expansion of a graph is quantified spectrally by smallest non-zero eigenvalue $\lambda_1$. For example, in communication networks and Markov chains to capture "bottlenecks" [27].

In Table I, smallest non-zero eigenvalues $\lambda_1$ are given for variation in geometry of the type depicted in Figure 3. For 3D problems (Figure 4), spectral gap $\Lambda$ comes out to be $\lambda_2 - \lambda_1$, then there is only one obstruction and $\lambda_1$ gives a measure of the size of narrow passage. Entries of the Table I capture $\lambda_1$ for different geometries (See Figure 3) and a revealing relationship between obstruction size and $\lambda_1$. Notice that $\lambda_1$ is significantly higher than zero for small or no obstructions thus revealing whether the underlying space has narrow passages or not. The more the distortion in the space, the lesser $\lambda_1$ until it falls to zero with a splitting of the space into two. The space is sampled with different number of samples. The higher the number of samples, the better the approximation of spectral gap and narrow passage.

3) Spectral Gap and Number of Obstructions: In case of multiple obstructions (Figure 4), spectral gap is $\Lambda = \lambda_{i+1} - \lambda_i$ for $i$ number of obstructions; which can be noted by the large difference (visible jump) in consecutive eigenvalues $\lambda_i$ and $\lambda_{i+1}$. (See Figure 5). We can get information about number of obstructions in any arbitrary space by noting the spectral gap $\Lambda$.

B. Eigenvectors and Expansion

Eigenvectors (Figures 3, 4) reveal important information about visibility, expansiveness and harmonic features of a space (number of obstructions/chambers). Eigenvectors are
an easier and direct method to get a measure of visibility and expansiveness. We can use other combinatorial methods to approximate expansiveness. One such example is considering the degree of each node.

1) Visibility of Nodes and Degree: In PRM graphs, visibility of a node can be defined as the degree of that node. These are plotted in Figure 6 for different spaces. It is observed that the nodes located near the neck have higher degrees. This is because nodes positioned at or near the neck have visibility to the other chamber as well. The higher the degree, the higher the visibility of the node configuration in both chambers and thus higher $\varepsilon$-goodness. If there are a larger number of such nodes having high visibility to the other chamber, it would mean that the configuration space is richly expansive. Minimum degree PRM graph approximately define $\varepsilon$-goodness of the configuration space. But this combinatorial method is not conclusive and specially in case of longer necks, where the degree of the nodes is not distinctly higher (Figure 6, top right). Therefore degree information alone is not sufficient for analyzing narrow passage and expansiveness.

2) Visibility to Other Chamber: To get the estimates of expansiveness ($\alpha, \beta$), degree information alone is not sufficient because degree does not distinguish between connections to different chambers. To first identify the various chambers, we use eigenvector $\psi_1$. It divides the space into two partitions, by distinguishing the nodes located in different chambers by different signs. If a node is connected to the node another node for which the signs of their eigenvector components disagree, then both nodes belong to different chambers. For each node, we use this information to count the number of connections to the nodes located in the other chamber. We plot this modified definition of a node’s degree (Figure 7) and note that some of the nodes are black. This indicates that such nodes have little or no connections to the other chamber. Nodes located close to the neck have higher values (visualized by a strong hue of red). It is noted that this modified degree is high even in the case of longer necks (Figure 7). Since, now we can check the visibility of a node to nodes of the other chamber, we can use this information to approximate the values of $\alpha$ and $\beta$.

Next we propose another method to estimate the visibility to the other chambers directly from the magnitudes of eigenvector components of $\psi_1$. Results verify this connection (Figure 8). We can see that essentially the visibility to the other chambers (from modified degree) and the magnitude of eigenvector $\psi_1$ (Figure 7 and 8) highlight the same nodes. Empirically, we can say that the magnitude of eigenvector $\psi_1$ gives us the same underlying information as visibility to the other chamber. Hence, eigenvector can be used to approximate parameters ($\alpha$ and $\beta$). The magnitude of eigenvector component of a node is a measure of its connection to the other chamber. The lower the magnitude, the larger the visibility to other chamber.

We can further elaborate this relationship by using a threshold $t$ on eigenvector magnitudes. For example in Figure 9, $t = 0.8$ removes nodes with poor visibility to other chamber (c.f. Figure 7, top left). We can tune this required visibility by varying threshold $t$. This variation gives different values of $\alpha$ and $\beta$. This can be observed in Figure 9.

3) Higher Harmonics and Configuration Space Features: In the case of a single obstruction (e.g. Figure 3 and 8), the eigenvector $\psi_1$ changes sign right at the neck, thus capturing the notion of a narrow passage spectrally. This is most discerning when the space is about to split and least informative in a perfect ball (convex shapes). The higher spectra do not reveal much additional information. This is also consistent with the fact that the space does not have any higher order features.

In a space with multiple chambers and connecting narrow passages (Figure 4), higher eigenvectors start to reveal a lot of useful information about the space. In particular, for the case of two obstructions (when spectral gap is $\lambda_3 - \lambda_2$), both $\psi_1$ and $\psi_2$ become interesting. The harmonics occupy higher spatial frequencies, exactly discerning the three chambers from each other. The trend continues for larger number of chambers. For a space with $i + 1$ chambers (i.e. spectral gap is $\lambda_{i+1} - \lambda_i$), $\psi_i$ partitions the nodes into $i + 1$ sets and
changes sign right at the necks. This is shown in Figure 4 for $\psi_3$ for three obstructions (bottom left) and $\psi_4$ for four obstructions (bottom right).

V. CONCLUSIONS AND FUTURE WORK

In this paper, we have presented a way to characterize the properties of the configuration space using spectral analysis on the PRM graph. It is evident that the spectral method is a useful, easily computable and visualizable alternative to direct geometrical methods of characterizing obstructions such as expansiveness. Eigenvalues and eigenvectors of the graph Laplacian reveal important information about the size and number of narrow passages and have a direct relationship with expansiveness. Particularly, the smallest non-zero eigenvalue detects the severity of configuration space distortion and the spectral gap helps identify the number of obstructions. Thus far we have only demonstrated an empirical relationship albeit very strong one between spectra and motion planning difficulties. Work in underway in making concrete analysis of these connections. Finally this work demonstrates the utility of spectral graph theory and spectral geometry methods in robot motion planning.

REFERENCES