Nonlinear Dynamics and Control of Mechanical Systems with Constraints and Symmetries: A Lagrangian Perspective

Abubakr Muhammad
Department of Electrical Engineering,
Georgia Institute of Technology.

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Contents

1 Introduction **3**

2 Some Mutli-variable calculus concepts in R^n **3**
2.1 Dual Space 3
2.2 Directional Derivatives 3

3 Differential Manifolds **3**
3.1 Manifold 3
3.2 Charts 3
3.3 Atlas 4
3.4 Mappings between Manifolds 4

4 Tangent and Cotangent Spaces **4**
4.0.1 $C^\infty(p)$ class of functions 4
4.1 Derivation 4
4.2 Tangent Space 4
4.3 Basis for Tangent Space 4
4.4 Tangent Map 5
4.5 Tangent Bundle 5
4.6 Projection of Tangent bundle on the manifold 5
4.7 Cotangent Spaces 5
4.8 Dual basis for Cotangent Spaces 5
4.9 Differentials as Cotangent Vectors for a given function 6
4.10 Cotangent Map 6
4.11 Cotangent Bundle 6

5 Vector Fields, Lie Derivatives and Lie Brackets **6**
5.1 Vector Fields 6
5.2 Differential Equations and Integral Curves 6
5.3 Flow of a vector Field 7
5.4 Lie Derivative 7

5.5 Fields, Flows and Lie Derivatives in R^n . . . 7
5.6 Motivation for Lie Brackets: Lie Derivatives don't commute . . . 7
5.7 Lie Bracket of two Vector Fields . . . 8
5.8 Properties of Lie Brackets . . . 9
5.9 Tangent Space of a Lie Bracket . . . 9
5.10 Affine Connections and Christoffel symbols . . . 9

6 Lie Groups and Lie Algebra **9**
6.1 Lie Groups . . . 10
6.2 Left and Right Translation . . . 10
6.3 Examples of Lie Groups . . . 10
6.4 Left and Right Invariant Vector Fields . . . 10
6.5 Lie Algebra . . . 11
6.6 Smooth Vector Fields on R^n . . . 11

7 The Frobenius Theorem **11**
7.1 Distributions . . . 11
7.2 Regularity . . . 11
7.3 Involutive Distribution . . . 11
7.4 Involutive closure . . . 12
7.5 Lie Algebra of the involutive closure . . . 12
7.6 Integrable Distribution . . . 12
7.7 Integral Manifolds . . . 12
7.8 Frobenius Theorem . . . 12
7.9 Example . . . 12

8 Description of Mechanical systems: A Lagrangian-Geometrical Perspective **13**
8.1 Configuration Spaces . . . 13
8.2 Lagrangian . . . 13
8.3 External Forces . . . 13
8.4 Equations of motion . . . 13
8.5 Constraints . . . 14
8.5.1 Holonomic Constraints . . . 14
8.5.2 Pfaffian constraints . . . 14
8.5.3 Lagrange-d'Alembert Equations . . . 14

9 Modelling Symmetries **14**
9.1 Actions of Lie Groups . . . 14
9.2 Orbits . . . 15
9.3 Lifted Action . . . 15
9.4 Infinitesimal Generator . . . 15
9.5 G-Invariance of the Lagrangian . . . 15
9.6 Tangent Space of the orbit . . . 15
9.7 Noether's Theorem and conservation laws . . . 15
9.8 Mechanical Connection . . . 16
9.9 Equations of motion with symmetry . . . 16

10 Literature Survey and references **16**

1 Introduction

In this report, we discuss some key ideas for modelling and control of nonlinear mechanical systems. The goal is to find a good description of the constraints and symmetries that are found in most systems of interest. All the necessary background from differential geometry is given. Main ideas discussed below are the celebrated Frobenius Theorem, description of Lagrangian dynamics using geometrical notions, and the equations of motion with constraints and symmetries.

2 Some Mutli-variable calculus concepts in R^n

2.1 Dual Space

For a linear vector space V, a *Dual Space* V^* is the space of *co-vectors* w such that $\langle w, v\rangle \in R$ for $v \in V$.

Examples

1. In R^n think of V as a space of column vectors and V^* as the row space.
2. For $f(x_1, x_2, \ldots, x_n)$ and $v \in R^n$,

$$\langle \frac{\partial f}{\partial x}, v\rangle = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} v_i \tag{1}$$

   Here $\frac{\partial f}{\partial x}$ is the covector of v.

2.2 Directional Derivatives

Given a function $f : R^n \to R$ the *differential df* is a *covector* field:

$$df = (\frac{\partial f}{\partial q^1}, \frac{\partial f}{\partial q^2}, \ldots, \frac{\partial f}{\partial q^n}) \tag{2}$$

If X is a *vector field* we can define the *directional derivative* as:

$$Ł_X f = \langle df, X\rangle \tag{3}$$

These concepts are now generalized below for differentiable manifolds.

3 Differential Manifolds

3.1 Manifold

A *manifold* of dimension n is a set M which is locally homeomorphic to R^n.

3.2 Charts

A manifold can be parameterized using *local coordinate charts*. A local coordinate chart is a pair (ϕ, U), where ϕ is a function that maps point in the set $U \subset M$ to an open subset of R^n. Two overlapping charts (ϕ, U) and (ψ, V) are C^∞ related if $\psi^{-1} \circ \phi$ is a diffeomorphism.

3.3 Atlas

A collection of such charts with the additional property that U's cover M, is a *smooth atlas*. If a manifold M admits a smooth atlas it is a *smooth manifold*.

3.4 Mappings between Manifolds

Let $F : M \to N$ be a mapping between two smooth manifolds M and N with coordinate charts (U, ϕ) and (V, ψ) respectively. The mapping F is smooth if $\tilde{F} = \psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth.

4 Tangent and Cotangent Spaces

4.0.1 $C^\infty(p)$ class of functions

It is the set of smooth real valued functions on M whose domain of definition includes some open neighborhood of $p \in M$

4.1 Derivation

A map $X_p : C^\infty(p) \to R$ is called a *derivation* if for all $\alpha, \beta \in R$ and $f, g \in C^\infty(p)$, it satisfies:

1. Linearity :
$$X_p(\alpha f + \beta g) = \alpha(X_p f) + \beta(X_p g) \tag{4}$$

2. Leibniz Rule:
$$X_p(fg) = (X_p f)g(p) + f(p)(X_p g) \tag{5}$$

The set of all derivations $X_p : C^\infty(p) \to R$ defines a vector space over the reals with the operations:

$$(X_p + Y_p)f = X_p f + Y_p f \tag{6}$$
$$(\alpha X_p)f = \alpha(X_p f) \tag{7}$$

4.2 Tangent Space

The tangent space T_pM of manifold M at point p, is the set of all derivations. Elements of a tangent space are called *tangent vectors*.

4.3 Basis for Tangent Space

The set of derivations $\{\frac{\partial}{\partial x_i}\}$ forms a basis for T_pM at the local coordinates $x = \phi(p) = (x_1, x_2, \ldots x_n)$ relative to the chart (ϕ, U). A derivation can therefore be written as:
$$X_p = X_1 \frac{\partial}{\partial x_1} + \ldots + X_n \frac{\partial}{\partial x_n} \tag{8}$$
where the vector $(X_1, X_2, \ldots X_n) \in R^n$ is a local representation of $X_p \in T_pM$.

4.4 Tangent Map

If $F : M \to N$ is a smooth map, then we can define the *tangent map* of F at p as the linear map $F_{*p} : T_pM \to T_{F(p)}N$ defined by:

$$F_{*p}X_p(f) = X_p(f \circ F) \tag{9}$$

where $X_p \in T_pM$ and $f \in C^\infty(F(p))$. We also sometimes write the tangent map of F at p as T_pF. The tangent map has the following properties:

1. If $H = F \circ G$ is a composition of two smooth maps, then $H_{*p} = F_{*G(p)} \circ G_{*p}$ or in the other notation as $T_pH = T_{G(p)}F \circ T_pG$
2. If $F : M \to N$ is a diffeomorphism then $F_{*p} : T_pM \to T_{F(p)}N$ is an isomorphism of tangent spaces with the inverse $(F_{*p})^{-1} = (F^{-1})_{*F(p)}$

4.5 Tangent Bundle

The *tangent bundle* of of a manifold M of dimension n, denoted by TM, is a manifold of dimension $2n$ defined by:

$$TM = \bigcup_{p \in M} T_pM \tag{10}$$

An element of TM is denoted by (p, X_p) where $p \in M$ and $X_p \in T_pM$.

4.6 Projection of Tangent bundle on the manifold

There is a natural projection $\pi : TM \to M$ given by:

$$\pi(X_p) = p \tag{11}$$

4.7 Cotangent Spaces

Generalizing the concept given above for dual linear vector spaces, for manifolds we have *Cotangent Spaces* T_p^*M. For a tangent space T_pM we define the cotangent space at p, as the set of all linear functions $\omega_p : T_pM \to R$. It has the same dimension as T_pM and elements of cotangent space are called *cotangent vectors*. The action of a cotangent vector $\omega \in T_p^*M$ on a tangent vector $X_p \in T_pM$ is written as $\langle \omega_p, X_p \rangle$.

4.8 Dual basis for Cotangent Spaces

If $\{\frac{\partial}{\partial x_i}\}$ is the basis for T_pM at local coordinates $(x_1, x_2, \ldots, x_n)$, then the *Dual Basis* for T_p^*M is given by $\{dx_i\}$, with the property that:

$$\langle dx_i, \frac{\partial}{\partial x_i} \rangle = \delta_{ij}, \quad i, j = 1 \ldots n \tag{12}$$

4.9 Differentials as Cotangent Vectors for a given function

When given a function, $f : M \rightarrow R$, the cotangent vector $df(p) \in T_P^*M$ is defined by:

$$\langle df(p), X_p \rangle = X_p(f), \quad X_p \in T_pM \tag{13}$$

Recognize $df(p)$ as the *differential* of f. For a given chart (ϕ, U) at local coordinates $x = \phi(p) = (x_1, x_2, \ldots x_n)$, $df(p)$ is give by:

$$df(x) = \frac{\partial f(x)}{\partial x_1} dx_1 + \ldots + \frac{\partial f(x)}{\partial x_n} dx_n \tag{14}$$

4.10 Cotangent Map

The *cotangent map* of a smooth map $F : M \rightarrow N$ is the linear map $F_p^* : T_{F(p)}^*N \rightarrow T_p^*M$ defined by:

$$\langle F_p^* \alpha_{F(p)}, X_p \rangle = \langle \alpha_{F(p)}, F_p^* X_p \rangle \tag{15}$$

where $\alpha_{F(p)} \in T_{F(p)}^*N$ and $X_p \in T_pM$. Like tangent maps, the cotangent maps also have an alternative notation T_p^*F.

4.11 Cotangent Bundle

The *cotangent bundle* is defined in the same way as the tangent bundle as a manifold T^*M of dimension $2n$,

$$T^*M = \bigcup_{p \in M} T_p^*M \tag{16}$$

5 Vector Fields, Lie Derivatives and Lie Brackets

5.1 Vector Fields

Vector fields represent differential equations on manifolds. For a coordinate chart (ϕ, U), a smooth vector filed $X : M \rightarrow TM$ can be written as:

$$X(x) = X_1(x)\frac{\partial}{\partial x_1} + \ldots + X_n(x)\frac{\partial}{\partial x_n} \tag{17}$$

where x = $\phi(p)$. Let $\aleph(M)$ denote the set of all smooth vector fields on M.

5.2 Differential Equations and Integral Curves

Vector fields represent differential equations on manifolds. Let $\gamma : (a, b) \rightarrow M$ be a curve on the manifold. The curve γ is said to be an *integral curve* of the vector field if

$$\dot{\gamma}(t) = X(\gamma(t)) \tag{18}$$

The existence and uniqueness of differential equations gaurantees the existence of the integral curves locally. The vector field is said to be *complete* if the domain of the curve is $(-\infty, \infty)$.

5.3 Flow of a vector Field

For a complete vector field, the integral curves define a one-parameter family of diffeomorphisms $\Phi_t(q) : M \to M$ where $\Phi_t(q)$ is the curve starting at $t = 0$ from the initial condition $q \in M$. This family of diffeomorphisms is called the *flow* of the vector field.

5.4 Lie Derivative

The Lie derivative of $f \in C^\infty(M)$ with respect to X is a new function $Xf : M \to R$ defined by:

$$Ł_X f = Xf(p) = X_p f \tag{19}$$

In coordinate chart (ϕ, U), at $x = \phi(p)$,

$$Ł_X f = Xf(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} X_i(x) = \langle df, X \rangle \tag{20}$$

where all partial derivatives are computed at $x = \phi(p)$.

5.5 Fields, Flows and Lie Derivatives in R^n

If we restrict our attention to R^n, then denoting $T_q R^n$, the tangent space to R^n, at point $q \in R^n$, a *vector field* on R^n is a smooth map which assigns to each point $q \in R^n$ a tangent vector $f(q) \in T_q R^n$. They can thus be thought of as right hand side of differential equations:

$$\dot{q} = f(q) \tag{21}$$

We define the *flow* of a vector field to represent the solution of the corresponding differential equation. $\Phi_t^f(q)$ represents the state of the differential equation at time t, starting from initial condition $q \in R^n$ at $t = 0$. Thus $\Phi_t^f(q) : R^n \to R^n$ satisfies

$$\frac{d}{dt}\Phi_t^f(q) = f(\Phi_t^f(q)) \tag{22}$$

Note that the flow satisfies the *group property*

$$\Phi_t^f \circ \Phi_s^f = \Phi_{t+s}^f \tag{23}$$

where the composition is defined as $\Phi_t^f(\Phi_s^f(q))$.

The time derivative of a smooth function $V : R^n \to R^n$ along f is the *Lie Derivative*

$$Ł_f V = \frac{\partial V}{\partial q} f(q) \tag{24}$$

5.6 Motivation for Lie Brackets: Lie Derivatives don't commute

Note that unlike ordinary derivatives, Lie derivatives don't commute. i.e. although,

$$\frac{\partial}{\partial q_i}\frac{\partial}{\partial q_j}V = \frac{\partial}{\partial q_j}\frac{\partial}{\partial q_i}V \quad (25)$$

$$\text{Ł}_f \text{Ł}_g V \neq \text{Ł}_g \text{Ł}_f V \quad (26)$$

Also the composition rule for vector flows is not commutative.

$$\Phi_t^f \circ \Phi_s^g \neq \Phi_s^g \circ \Phi_t^f \quad (27)$$

Let us explore this a bit more. Let a flow starting from q_0 be a composition $\Phi_\epsilon^{-g_2} \circ \Phi_\epsilon^{-g_1} \circ \Phi_\epsilon^{g_2} \circ \Phi_\epsilon^{g_1}$. i.e. a flow of ϵ seconds along g_1, then ϵ seconds along g_2, then ϵ seconds along $-g_1$ and finally ϵ seconds along $-g_2$. If the commutation were true we would have a net flow of zero, however this is not true as seen below.

$$\begin{aligned}
q(\epsilon) &= \Phi_\epsilon^{g_1}(q(0)) && (28)\\
&= q(0) + \epsilon\dot{q}(0) + \frac{\epsilon^2}{2}\ddot{q}(0) + O(\epsilon^3) && (29)\\
&= q_0 + \epsilon g_1(q_0) + \frac{\epsilon^2}{2}\frac{\partial g_1}{\partial q}g_1(q_0) + O(\epsilon^3) && (30)\\
q(2\epsilon) &= \Phi_\epsilon^{g_2} \circ \Phi_\epsilon^{g_1}(q_0) && (31)\\
&= \Phi_\epsilon^{g_2}(q_0 + \epsilon g_1(q_0) + \frac{\epsilon^2}{2}\frac{\partial g_1}{\partial q}g_1(q_0) + O(\epsilon^3)) && (32)\\
&= q_0 + \epsilon(g_1(q_0) + g_2(q_0)) + \frac{\epsilon^2}{2}(\frac{\partial g_1}{\partial q}g_1(q_0) + \frac{\partial g_2}{\partial q}g_2(q_0) + 2\frac{\partial g_2}{\partial q}g_1(q_0)) + O(\epsilon^3) && (33)\\
q(3\epsilon) &= \Phi_\epsilon^{-g_1}\Phi_\epsilon^{g_2} \circ \Phi_\epsilon^{g_1}(q_0) && (34)\\
&= q_0 + \epsilon g_2(q_0) + \frac{\epsilon^2}{2}(\frac{\partial g_2}{\partial q}g_2(q_0) + 2\frac{\partial g_2}{\partial q}g_1(q_0) - 2\frac{\partial g_1}{\partial q}g_2(q_0)) + O(\epsilon^3) && (35)\\
q(4\epsilon) &= \Phi_\epsilon^{-g_1}\Phi_\epsilon^{g_2} \circ \Phi_\epsilon^{g_1}(q_0) && (36)\\
&= q_0 + \epsilon^2(\frac{\partial g_2}{\partial q}g_1(q_0) - \frac{\partial g_1}{\partial q}g_2(q_0)) + O(\epsilon^3) && (37)\\
& && (38)
\end{aligned}$$

Note that the net result of flow around this "square" defined by the two vector fields is the infinitesimal motion of order ϵ^2. We call this "net motion" the Lie Bracket.

5.7 Lie Bracket of two Vector Fields

We define the *Lie Bracket* of two vector fields g_1 and g_2 as:

$$[g_1, g_2](q) = \frac{\partial g_2}{\partial q}g_1(q) - \frac{\partial g_1}{\partial q}g_2(q) \quad (39)$$

The Lie Bracket of two vector fields satisfies the following for all smooth functions $\alpha : R^n \to R^n$,

$$\text{Ł}_{[g_1,g_2]}\alpha = \text{Ł}_{g_1}(\text{Ł}_{g_2}\alpha) - \text{Ł}_{g_2}(\text{Ł}_{g_1}\alpha) \quad (40)$$

For example, the Lie Bracket of two linear vector fields $f(q) = Aq$ and $g(q) = Bq$ is given by:

$$[f, g](q) = (BA - AB)q \tag{41}$$

Note that this is itself a linear vector field.

If $[f, g] = 0$ then f and g are said to *commute*.

5.8 Properties of Lie Brackets

Given smooth maps $\alpha, \beta : R^n \to R$, and vector fields, f, g, h on R^n, the Lie Bracket satisfies the following properties.

1. *Skew Symmetry*: $[f, g] = -[g, f]$.
2. *Jacobi's Identity*: $[f, [g, h]] + [h, [f, g]] + [g, [h, f]] = 0$.
3. *Chain Rule*: $[\alpha f, \beta g] = \alpha\beta[f, g] + \alpha(Ł_f \beta)g - \beta(Ł_g \alpha)f$

5.9 Tangent Space of a Lie Bracket

If X and Y are smooth vector fields on M and F is a smooth map then,

$$F_*[X, Y] = [F_*X, F_*Y] \tag{42}$$

5.10 Affine Connections and Christoffel symbols

An affine connection ∇, maps two vector fields X and Y into a third vector field $\nabla_X Y$ satisfying the following properties.

1. $\nabla_{\alpha X} Y = \alpha \nabla_X Y$
2. $\nabla_X \alpha Y = (Ł_X \alpha) Y + \alpha \nabla_X Y$

∇ determines, and is uniquely determined by the Christoffel symbols.

$$\nabla_{\frac{\partial}{\partial q_i}} \frac{\partial}{\partial q_j} = \Gamma^k_{ij} \frac{\partial}{\partial q_k} \tag{43}$$

In terms of individual coordinates the affine connection can be written as:

$$\begin{aligned} \nabla_X Y &= (Ł_X Y_k + \Gamma^k_{ij} X_i Y_j) \frac{\partial}{\partial q_k} && (44) \\ &= (\frac{\partial Y_k}{\partial q_i} X_i + \Gamma^k_{ij} X_i Y_j) \frac{\partial}{\partial q_k} && (45) \end{aligned}$$

6 Lie Groups and Lie Algebra

We discuss below some definitions and properties of Lie Algebras and groups that are extensively used in controls and robotics for the formulation of symmetries.

6.1 Lie Groups

A *Lie group* is a group which is also a smooth manifold and for which the group operations $(g_1, g_2) \mapsto g_1 g_2$ and $g \mapsto g^{-1}$ are smooth. A Lie Group is *Abelian* if $gh = hg, \forall g, h \in G$.

6.2 Left and Right Translation

For every $g \in G$ we define the *Left Translation* $L_g : G \to G$ given by $L_g(h) = gh$ for $h \in G$. Similarly *Right Translation* is the map $R_g : G \to G$ satisfying $R_g(h) = hg$.

Since $L_g \circ L_h = L_{gh}$ and $R_g \circ R_h = R_{gh}$, we have that $(L_g)^{-1} = L_{g^{-1}}$ and $(R_g)^{-1} = R_{g^{-1}}$. Thus both L_g and R_g are diffeomorphisms of G for each g. Moreover left and right translation commute: $L_g \circ R_h = R_h \circ L_g$. If the group is abelian then $L_g = R_g$.

6.3 Examples of Lie Groups

1. $GL(n, R)$ The group of all $n \times n$ nonsingular matrices. As a manifold it is an open subset of R^{n^2}. The group operation is matrix multiplication, inversion is matrix inversion, left and right translations are left and right multiplications respectively. Note that both inversion and product are smooth.

2. $SO(n)$ The special Orthogonal group. It is a subgroup of $GL(n, R)$ defined as $SO(n) = \{R \in GL(n, R) : RR^T = I, det(R) + 1\}$. The dimension of manifold is $n(n-1)/2$.

3. $SE(3)$. Special Euclidean group. It is the group of rigid transformations of the form $g(x) = Rx + p$ where $R \in SO(3)$ and $p \in R^3$. $SE(3)$ can be identified with the space of 4×4 matrices of the form:

$$g = \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} \tag{46}$$

   $SE(3)$ is a Lie Group of dimension 6.

6.4 Left and Right Invariant Vector Fields

A vector field Y over a Lie group G is *left-invariant* if

$$T_g L_h Y(g) = Y(hg), \quad \forall h, g \in G \tag{47}$$

where $T_g L_h$ is the linearization of L_h at g, (recall $T_g F$ is the tangent map of a smooth map F at g). Similarly, the vector field Y is *right-invariant* if,

$$T_g R_h Y(g) = Y(gh), \quad \forall h, g \in G \tag{48}$$

An alternative notation is to write the left invariance as:

$$(L_h)_* X = X \tag{49}$$

and the right invariance as:

$$(R_h)_* X = X \tag{50}$$

6.5 Lie Algebra

A vector space V is a *Lie Algebra* if there exists a bilinear operator $V \times V \to V$ denoted by $[.,.]$ satisfying:

1. *Skew Symmetry*: $[v, w] = -[w, v]$.
2. *Jacobi's Identity*: $[v, [w, z]] + [z, [v, w]] + [w, [z, v]] = 0$.

for $v, w, z \in V$. A subspace $W \subset V$ is called a *Lie subalgebra* if $[v, w] \in W$ for all $v, w \in W$.

6.6 Smooth Vector Fields on R^n

The set of smooth vector fields on R^n with the Lie bracket is a Lie algebra, denoted by $\aleph(R^n)$.

Let $\aleph_L(G)$ denote the set of left invariant vector fields on G, then for $X, Y \in \aleph_L(G)$, we have:

$$L_{h*}[X, Y] = [L_{h*}X, L_{h*}Y] = [X, Y] \tag{51}$$

Thus $\aleph_L(G)$ is a Lie subalgebra of the Lie algebra $\aleph(G)$.

7 The Frobenius Theorem

7.1 Distributions

A *distribution* assigns a subspace of the tangent space to each point in R^n in a smooth way. A special case is a distribution defined by a set of smooth vector fields $g_1, g_2, \ldots, g_m$. In this case we define the distribution as:

$$\Delta = span\{g_1, g_2, \ldots, g_m\} \tag{52}$$

where we take the span over the set of smooth real-valued function on R^n. At $q \in R^n$, the distribution defines a linear subspace of the tangent space:

$$\Delta_q = span\{g_1(q), g_2(q), \ldots, g_m(q)\} \subset T_q R^n \tag{53}$$

7.2 Regularity

A distribution is said to be *regular* if the dimension of the subspace Δ_q does not vary with q.

7.3 Involutive Distribution

A distribution is involutive if it is closed under the Lie bracket, i.e. Δ is involutive if and only if $\forall f, g \in \Delta$, $[f, g] \in \Delta$.

For a finite dimensional distribution, this can be checked by verifying that Lie bracket of the basis elements are contained in the distribution. In other words, a linearly independent set of vector fields, $\{g_1, g_2, \ldots, g_n\}$ is said to be involutive if and only if there exist scalar functions $\alpha_{ijk} : R^n \to R$ such that,

$$[g_i, g_j] = \sum_{k=1}^{m} \alpha_{ijk}(x) g_k(x), \quad \forall i, j \tag{54}$$

7.4 Involutive closure

The *involutive closure* $\bar{\Delta}$ is the smallest distribution containing Δ such that if $f, g \in \bar{\Delta}$ then $[f, g] \in \bar{\Delta}$. It is the closure of Δ under bracketing.

7.5 Lie Algebra of the involutive closure

Note that the involutive closure $\bar{\Delta}$ qualifies as a Lie Algebra. It is the smallest Lie algebra containing $g_1, g_2, \ldots g_m$. It is called the Lie algebra *generated* by $g_1, g_2, \ldots g_m$ and is denoted by $L(g_1, g_2, \ldots g_m)$. The elements of this Lie algebra are obtained by taking all linear combinations of $g_1, g_2, \ldots g_m$, taking Lie brackets of these, taking linear combinations and so on. The *rank* of $L(g_1, g_2, \ldots g_m)$ at point $q \in R^n$ is the dimension of $\bar{\Delta}_q$ as a distribution.

7.6 Integrable Distribution

A distribution Δ of a constant dimension k, is said to be *integral* if for every point $q \in R^n$, there exists a set of smooth functions $h_i : R^n \to R$, $i = 1, \ldots, n-k$ such that the row vectors $\frac{\partial h_i}{\partial q}$ are linearly independent at q and for every $f \in \Delta$,

$$L_f h_i = \frac{\partial h_i}{\partial q} f(q) = 0, \quad i = 1, \ldots, n-k \tag{55}$$

7.7 Integral Manifolds

The hypersurfaces defined by the level sets $\{q : h_i(q) = c_i, i = 1 \ldots n-k\}$ are called *Integral Manifolds* for the distribution.

7.8 Frobenius Theorem

A regular distribution is integrable if and only if it is involutive.

7.9 Example

Consider the following set of partial differential equations:

$$4x_3 \frac{\partial h}{\partial x_1} - \frac{\partial h}{\partial x_2} = 0 \tag{56}$$

$$-x_1 \frac{\partial h}{\partial x_1} + (x_3^2 - 3x_2)\frac{\partial h}{\partial x_2} + 2x_3 \frac{\partial h}{\partial x_3} = 0 \tag{57}$$

The vector fields are $g_1 = [4x_3, -1, 0]^T$ and $g_2 = [-x_1, (x_3^2 - 3x_2), 2x_3]^T$. The Lie bracket of the two fields is computed below,

$$\begin{aligned} [g_1, g_2] &= \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2 \qquad (58) \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 2x_3 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -4x_3 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -x_1 \\ x_3^2 - 3x_2 \\ 2x_3 \end{pmatrix} \end{aligned}$$

$$[g_1, g_2] = \begin{pmatrix} 12x_3 \\ 3 \\ 0 \end{pmatrix} = -3g_1 + 0g_2$$

This shows that the vector fields are involutive and therefore by Frobenius's theorem, they are solvable.

8 Description of Mechanical systems: A Lagrangian-Geometrical Perspective

Given below is a formulation of Lagrangian Mechanics based on the ideas of differential geometry discussed above.

8.1 Configuration Spaces

Let Q be a smooth (C^∞) manifold of dimension n, then the configuration q of a mechanical system can be described as a point on the manifold Q. The tangent space at the configuration $q \in Q$ is T_qQ, the tangent bundle is TQ and the cotangent bundle as T^*Q.

8.2 Lagrangian

A mechanical system on Q is described by the Lagrangian $L : TQ \to R$. This description describes the Lagrangian as a mapping from the tangent bundle of the manifold to the space of real numbers, and is usually assumed to have a form of kinetic energy minus potential energy. If $(q, v) \in T_qQ$, represents a point in the bundle, then,

$$L(q, v) = \frac{1}{2} v^T M(q) v - V(q) \tag{59}$$

where $M(q)$ is the inertia matrix for the system.

8.3 External Forces

The external forces lie on the cotangent bundle T^*Q of the manifold. We let $F \in T^*Q$ represent the forces on the system and they include dissipation and actuator forces. If these forces are allowed to depend on the current configuration and velocity of the system we can define the forces as maps $F : TQ \to T^*Q$.

8.4 Equations of motion

The equations of motion for a Lagrangian System, with Lagrangian $L : TQ \to R$ and $F : TQ \to T^*Q$ are given by:

$$\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \right) \delta q^i = F_i(q, v) \delta q^i, \qquad \delta q \in T_qQ \tag{60}$$

In terms of intertia matrix the dynamics can be written as:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \frac{\partial V}{\partial q} = F \tag{61}$$

where $C(q,\dot{q})$ is the Coriolis matrix given by:

$$C_{ij}(q,\dot{q}) = \left(\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{kj}}{\partial q_i} \right) \dot{q}_k \tag{62}$$

and $F \in R^n$ is the vector of external forces. Note that if we denote the constraint forces by $F_c \in T^{\infty}Q$, i.e. forces that force the system to evolve on Q, satisfy $\langle F_c, \delta q \rangle = 0$ for all variations $\delta q \in T_qQ$. This is always the case when we have holonomic constraints. The configuration space is the level set of an algebraic relation on a higher dimensional manifold.

8.5 Constraints

Constraints can be modelled in two ways:

8.5.1 Holonomic Constraints

When the dynamics of the system are restricted to a manifold Q, are given by the level set of an algebraic constraint on the system configuration, the constraints are said to be holonomic.

8.5.2 Pfaffian constraints

If there are velocity constraints of the form

$$\langle \omega^j(q), \dot{q} \rangle = \omega_i^j(q)\dot{q^i} = 0, \qquad j = 1, \ldots k \tag{63}$$

the constraints are called Pfaffian. Here $\omega^j(q) \in T_q^*Q$ and are smooth and linearly independent.

8.5.3 Lagrange-d'Alembert Equations

The d'Alembert principle can be stated as: The forces of constraints satisfy $\langle F_c, \delta q \rangle = 0$ for all δq such that $\langle \omega^j(q), \dot{q} \rangle = 0$ for $j = 1, \ldots, k$. The equations of motion for a Lagrangian System under Pfaffian constraints are given by:

$$\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \right) \delta q^i = F_i(q,v)\delta q^i, \qquad \delta q \in D_q \tag{64}$$

$$\langle \omega^j(q), \dot{q} \rangle = 0, \qquad j = 1, \ldots k \tag{65}$$

where $D_q = \{v_q : \langle \omega^j(q), \dot{q} \rangle = 0, j = 1, \ldots k\}$

9 Modelling Symmetries

9.1 Actions of Lie Groups

Let M be a smooth manifold and G be a Lie group. A *Left action* of G on M, is a smooth map $\Phi : G \times M \to M$, (also written as Φ_g) such that

1. $\Phi(e, x) = \Phi_e(x) = x$ for all $x \in M$

2. For every $g, h \in G$ and $x \in M$, $\Phi(g, \Phi(h, x)) = \Phi_g(\Phi_h(x)) = \Phi_{gh}(x) = \Phi(gh, x)$

9.2 Orbits

If Φ is an action of G on M and $x \in M$, the orbit of x is defined by:

$$Orb_x = \{\Phi_g(x) : g \in G\} \tag{66}$$

9.3 Lifted Action

Let $\Phi_g : M \to M$ be an action of G on M. One can lift this action to an action on the tangent bundle TM, by $\Phi_* : G \times TM \to TM$ or in another notation as $\Phi_{*g} : TM \to TM$, defined by:

$$\Phi_{*g}(v_x) = T_x\Phi_g.v_x \tag{67}$$

9.4 Infinitesimal Generator

Let G be a matrix Lie Group and let $\Phi_g : Q \to Q$ be a free left action of G on Q. We write $\Phi_{g*} : TQ \to TQ$ for the lifted action of G on TQ and $\xi_Q : Q \to TQ$ for the infinitesimal generator defined by :

$$\xi_Q(q) = \frac{d}{ds}\Phi_{exp(\xi s)}(q)|_{s=0}, \qquad \xi \in g \tag{68}$$

where g is the Lie Algebra of G and $exp : g \to G$ is the exponential map.

9.5 G-Invariance of the Lagrangian

A Lagrangian $L : TQ \to R$ is G-invariant if:

$$L(\Phi_g(q), \Phi_{g*}(v)) = L(q, v), \qquad \forall g \in G \tag{69}$$

9.6 Tangent Space of the orbit

We write $Orb(q)$ for the orbit of a point q under the action of G and T_qOrb for the tangent space of the orbit at a point q. It follows from these definitions and definition of ξ_Q that $T_qOrb(q) = \{\xi_Q(q) : \xi \in g\}$

9.7 Noether's Theorem and conservation laws

For unconstrained systems, (or holonomically constrained systems) the presence of a symmetry implies the existence of a conservative law of the form:

$$\frac{d}{dt}\langle\frac{\partial L}{\partial \dot{q}}, \xi_Q\rangle = 0, \qquad \xi \in g \tag{70}$$

The most common examples are conservation of linear and angular momentum, corresponding to $G = (R^3, +)$ and $G = SO(3)$. It is convenient to re-write the conservation law in terms of the momentum map, $J : TQ \to g^*$,

$$\langle J(q,\dot{q}),\xi\rangle = \langle\frac{\partial L}{\partial \dot{q}},\xi_Q\rangle = \mu \tag{71}$$

where μ is a constant. The equation is is interpreted as the momentum in the ξ_Q direction being constant along the flows of the system.

9.8 Mechanical Connection

By making use of conservation of momentum, it is possible to reduce the description of the dynamics to a lower dimensional space. Define the *locked inertia tensor* $I(q) : g \to g^*$ as the metric that satisfies

$$\langle I\eta\rangle = \langle\langle \eta_Q, \xi_Q\rangle\rangle \tag{72}$$

where $\langle\langle .,.\rangle\rangle$ is the Reimannian metric given by the kinetic energy. We use this tensor to define the *Mechanical Connection* $A : TQ \to g$,

$$A = I^{-1}.J \tag{73}$$

The mechanical connection acts as a projection of the tangent space T_qQ onto the Lie Algebra g:

$$A(\xi_Q) = \xi \tag{74}$$

Thus given a velocity vector $v_q \in T_qQ$ the mechanical connection provides a means of splitting v_q into a *vertical part* $ver(v_q) = (A(v_q))_Q \in T_qOrb$ and *horizontal part* $hor(v_q)$ which satisfies $A(hor(v_q)) = 0$.

9.9 Equations of motion with symmetry

Using mechanical connection, the equations of motion for an unforced Lagrangian system with symmetry can be written in the following form:

$$\begin{aligned} A(q)\dot{q} &= I^{-1}(q)\mu && (75)\\ \dot{\mu} &= 0 && (76)\\ \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}\right)\delta q^i &= 0, \qquad A(q)\delta q = 0 && (77) \end{aligned}$$

These equations divide the dynamics into a first order set of equations in the vertical direction, a momentum equation in the form of a conservation law, and a set of second order equations in the horizontal direction.

10 Literature Survey and references

1. Mathematical Preliminaries
   (a) Differential Manifolds
   (b) Tangent and Cotangent Spaces
   (c) Vector Fields and Lie Brackets

(d) Lie Groups and Lie Algebra

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2. Lagrangian Mechanics: A Nonlinear Controls Perspective

(a) Langrangian Mechanics using Differential Geometry concepts

(b) Symmetries and Contraints

(c) Controllability

(d) Lie Brackets and Frobenius Theorem

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3. Examples and Case Studies

(a) Rigid Body Dynamics on Euclidean Group

(b) Dynamics of Robotics Manipulators

(c) Control Of Constrained Manipulators

(d) Multifingered Hands and Coordinated control

(e) Dynamics of Multibody Vehicles

(f) Human Locomotion

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4. Mechanics and Symmetry

(a) Non-holonomic constraints and Symmetries

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