Detecting Narrow Passages in Configuration Spaces Via Spectra of Probabilistic Roadmaps

Hassan Mohy-ud-Din*
School of Science & Engineering
Lahore University of Management Sciences
Lahore, Pakistan
mohyuddin.engineer@gmail.com

Abubakr Muhammad †
School of Science & Engineering
Lahore University of Management Sciences
Lahore, Pakistan
abubakr@lums.edu.pk

ABSTRACT

In this paper, we explore the connection between the spectral properties of a configuration space with those of the underlying probabilistic road map. We explore this relationship in a simple motion planning example which leads to a new method of characterizing narrow passages using the so called graph Laplacian of the PRM.

Categories and Subject Descriptors

I.2.9 [**Robotics**]: Workcell organization and planning; I.2.8 [**Problem Solving, Control Methods, and Search**]: Plan execution, formation, and generation

Keywords

Spectral analysis, motion planning, PRM, eigenvalues and eigenvectors, graph Laplacian, algebraic connectivity, configuration spaces, motion planning, robotics

1. INTRODUCTION

This paper presents a spectral analysis of the Probabilistic Roadmap (PRM) algorithm for motion planning in robotics [1, 11, 13]. PRM is a an extremely effective motion planning technique, that captures the connectivity of a high-dimensional configuration space $\mathcal{C}_{\text{space}}$, cluttered with obstacles and narrow passages by a graph of 1-D curves i.e. a roadmap [4, 5, 6, 9].

Section 2 begins by describing the PRM as a path planning approach to connect source and goal configurations of a robot characterized by many degrees of freedom [4, 5]. It parameterizes PRM by the visibility property of expansiveness of the free space $\mathcal{C}_{\text{free}}$ [6]. It is followed by addressing the more geometrically complex $\mathcal{C}_{\text{space}}$ containing narrow passages. Finally, the Eigenvalues and Laplacian of the constructed roadmap are studied with an emphasis on algebraic connectivity of $\mathcal{C}_{\text{free}}$. Section 3 begins by giving a short summary of various results of spectral geometry for Riemannian manifolds and Euclidean spaces. Next, the spectra of graphs is presented thus making a connection with the PRM algorithm. Section 4 is concluded with a summary of the results achieved and the future work that can done in this realm.

*Currently with the department of electrical & computer engineering, Johns Hopkins University, MD., USA.

†This work has been done at the Laboratory for Cyber-Physical Networks and Systems (CyPhyNetS), funded by SSE-LUMS via a faculty startup grant. http://cyphynets.lums.edu.pk/

2. MOTION PLANNING USING PRM

Probabilistic Roadmap (PRM) is a landmark path planning algorithm that overcomes the computational complexity and staggering costs of acquiring an exact representation of a high-dimensional configuration space, $\mathcal{C}_{\text{space}}$ [9]. PRM planning algorithm is a two phase process: *learning* and *query* [4]. In the learning phase, a PRM is generated by completely randomized sampling [14, 15] of collision-free configurations called milestones, from a high-dimensional $\mathcal{C}_{\text{space}}$, and connecting them via a fast, simple and deterministic motion planner called a local planner [4, 9]. The constructed roadmap (1D-curves) has two sets of parameters i.e. vertices and edges. Vertices, V, or nodes refer to the sampled milestones and edges, E, show a collision-free path connecting the two milestones. This roadmap shows the connectivity of $\mathcal{C}_{\text{free}}$.

In the query phase, the source and goal configurations are added to the PRM and the roadmap is searched for the shortest path, connecting the two query configurations, by Dijkstra's Algorithm. For PRM to be probabilistically complete, $\mathcal{C}_{\text{free}}$ must satisfy visibility properties of expansiveness [6, 9, 16]. The definition of expansiveness of a free space, $\mathcal{C}_{\text{free}} = F$, is as follows:

Definition 2.1. *Let α, β and ϵ be constants in the open interval $(0, 1)$. The free space, F , is $(\alpha, \beta, \epsilon)-$expansive if each of its connected components $F' \subseteq F$ satisfies the following conditions.*

1. *For every point $p \in F'$, $\mu(V(p)) \geq \epsilon \times \mu(F)$ i.e. every point is $\epsilon -$ good.*

2. *For any connected subset $S \subseteq F'$, the set:*

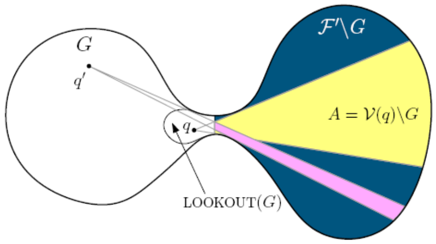

Figure 1: LOOKOUT and expansiveness, as defined in [5].

$$\beta\text{−}LOOKOUT(S) = \{q \in S|\mu(V(q)\backslash S) \geq \beta \times \mu(F'\backslash S)\}$$

has volume $\mu(\beta\text{-}LOOKOUT(S)) \geq \alpha \times \mu(S)$

where $\mu(S)$ *is the volume of set* S, $\mu(F) = 1$ *and* $V(q)$ *is the visibility set of the configuration* $q \subseteq F$. *In other words,* $V(q) = \{q' \in F|Freepath(q, q') is\ true\}$

For an (α, β, ε)-expansive free space F, the probability of failure of the PRM planner has an upper bound given by

$$Pr < \frac{c_1}{\varepsilon\alpha} \exp\left(c_2\varepsilon\alpha(-N + \frac{c_3}{\beta})\right)$$

where c_1, c_2 and c_3 are positive constants and N is the number of sampled milestones [9].

This upper-bound states that as the number of sampled milestones, N, increases the failure probability of PRM planner exponentially approaches zero.

Parameters α and β dictate the extent to which the visibility region of the sampled configurations can be expanded. For a configuration space, C_{Space}, characterized by high values of α and β, PRM easily captures the connectivity of the space by a small roadmap (i.e. smaller N) [6, 9].

Computational complexity varies proportionally with the geometric complexity of the C_{Space} which in turn affects the performance of PRM planner [9]. The geometrical complexity is affected by the degrees of freedom of the C_{Space} and the magnitude of clutter present in it [9, 11]. One of the geometric feature is the presence of narrow passages. Existence of narrow passages in C_{Space} is deliberate and at the same time in-advertent [9]. Deliberate in the case of alpha puzzle (6-D C_{Space}), a motion planning problem, in which the aim is to separate the intertwined tubes. In-advertent in the case when the robot has to traverse between closely spaced obstacles e.g. Bug Trap or when two mechanical components (CAD Designing) need to be inserted into each other e.g. Flange Problem [25].

C_{Space} containing narrow passages exhibit poor expansiveness, characterized by low values of (α, β, ε). PRM performance suffers dramatically in poorly expansive environments and calls for a high sampling density of milestones in such regions [6, 7]. Sampling in narrow passages carries low probability because of small visibility sets. Therefore, there is an inevitable need of smart algorithms to locate narrow passages in C_{Space} and sample highly in their neighborhood [10]. Algorithms like the Bridge Test, Small Step Retraction Method and Free Space Dilation sample narrow passages in PRM planning [8, 10, 12].

3. SPECTRA OF CONFIGURATION SPACES & PRMS

In this section, we make a connection between the spectral geometry of the underlying continuous configuration spaces and the spectrum of the roadmap graph. These concepts originate from classical subjects of spectral geometry and algebraic graph theory, driven by the fundamental question: *Can you hear the shape of a space* [20]?

3.1 Spectral Geometry of Euclidean Spaces and Manifolds

Since many (perhaps most) robotic configuration spaces can be modeled as Riemannian manifolds [19, 18], we start by describing the spectral geometry of such manifolds. For any $C^k, k \geq 2$ function on a manifold M, we define the Laplacian of f as

$$\Delta f = \text{div}(\text{grad } f).$$

If the Riemannian metric $G = (g_{ij})$ on M is given by

$$g_{ij} = \langle \partial_i, \partial_j \rangle, \quad G^{-1} = (g^{ij}),$$

where vector fields ∂_i denote the basis of the tangent space, then the Laplacian has an explicit formula given by

$$\Delta f = \frac{1}{\sqrt{g}} \sum_{j,k} \partial_j \left(g^{jk} \sqrt{g} \partial_k f \right).$$

Here $g = \det(G)$. For Euclidean spaces $G = I_n$, and the formula simplifies to a more familiar form. E.g. the planar Laplacian is given by

$$\Delta f = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f.$$

The eigenvalue problem for a Laplacian on a given M is to find all numbers λ, for which there exist a nontrivial solution $\phi \in C^2(M)$ to

$$\Delta\phi = -\lambda\phi.$$

Variants of this problem are known as The Neumann and Dirichlet eigenvalue problems [19]. It is well-known that the set of these eigenvalues for a given M consist of a sequence

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots,$$

and each associated eigenspace is finite dimensional. Let $N(\lambda)$ be the number of eigenvalues counted with multiplicity $\leq \lambda$, then

$$N(\lambda) \sim \text{vol}(M)\text{vol}(B_1^n) \frac{\lambda^{n/2}}{(2\pi^n)},$$

where B_1^n is the volume of of a unit ball in $\mathbb{R}^n$. This is the famous Weyl's asymptotic formula that started the field of spectral geometry. Since the discovery of this result, numerous estimates have been produced on the eigenvalues of the Laplacian and their geometric consequences. Of special interest is the smallest non-zero eigenvalue, λ_1, also known as the spectral gap. Consider a domain $\Omega \subset \mathbb{R}^n$ with $\mathrm{vol}(\Omega) = \mathrm{vol}(B_1^n)$. Then Rayleigh's conjecture [19] (proven in 1923 by Faber and Krahn) says that

$$\lambda_1(\Omega) \leq \lambda_1(B_1^n), \tag{1}$$

with equality if and only if $\Omega = B_1^n$. Similarly, the Payne-Poyla-Weinberger conjecture [19] (proven in 1991), says that

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(B_1^n)}{\lambda_1(B_1^n)}.$$

Finally, if κ denotes the sectional curvature [18] of $\Omega \subseteq M$ and M is simply connected, then

$$\lambda_1(\Omega) \geq -(n-1)^2\kappa/4. \tag{2}$$

This inequality combined with Eq. 1 is specially interesting when $\kappa < 0$ [19]. It provides a lower bound on λ_1. The more negatively curved the space, the higher the spectral gap λ_1. From Eq. 1 we can immediately see that this will have consequences for characterizing narrow passages in a configuration space. The further a space is from a regular ball of constant radius, the more possibility it has of having a narrow neck and the more difficult the motion planning problem. For computational purposes, instead of doing our spectral analysis directly onto the continuous configuration spaces, we resort to a discretization of our Laplacian which directly connects us to the PRM algorithm.

3.2 Graph Laplacians and Spectra

The PRM graph (G) can be parameterized by an ordered pair of *Vertices (V)* and *Edges (E)* i.e. $G = (V, E)$: where V refers to the sampled milestones and E refers to a collision-free path that exists between the two vertices.

The algebraic interpretation of $G = (V, E)$ is an *Adjacency Matrix*. Adjacency matrix, $A := (a_{i,j})_{n \times n}$ of a graph, with no self-loops, is defined as:

$$a_{i,j} := \begin{cases} 0 & : i = j \\ 1 & : i \neq j \textit{ and } v_i,\ v_j \textit{ are adjacent} \\ 0 & : \textit{otherwise} \end{cases}$$

Another matrix representation of graph is the *Laplacian matrix* or the *Admittance matrix* or the *Kirchhoff matrix*. The Laplacian matrix, $L := (l_{i,j})_{n \times n}$, of a graph, with no self-loops, is defined by:

$$l_{i,j} := \begin{cases} deg(v_i) & : i = j \\ -1 & : i \neq j \textit{ and } v_i,\ v_j \textit{ are adjacent} \\ 0 & : \textit{otherwise} \end{cases}$$

The Laplacian matrix of a graph is the difference between the Degree and the Adjacency Matrices i.e. $L := D - A$. The degree matrix, $D := (d_{i,j})_{n \times n}$, of a graph, with no self-loops, is defined by:

$$d_{i,j} := \begin{cases} deg(v_i) & : \text{if } i = j \\ 0 & : \textit{otherwise} \end{cases}$$

where the *degree* of a vertex is the number of vertices it connects to, i.e.

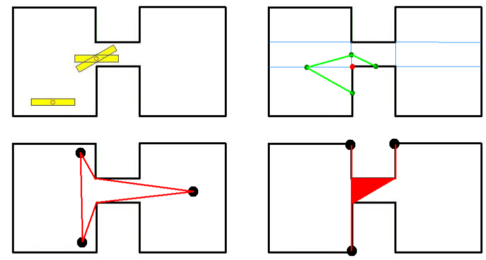

Figure 2: [top left] A motion planning example. [top right] Cubical complex of the free space. [bottom left] Geodesic triangle demonstrating negative curvature [bottom right] Another extreme example of a geodesic triangle.

$$deg(v_i) := \#(0 \leq j \leq n-1 : \forall j, a_{i,j} = 1)$$

Incidentally, this graph Laplacian is a direct discretization of its continuous analog [21, 22, 23, 17]. Laplacian matrix reveals insightful information about the graph. The *eigenvector decomposition* of the Laplacian matrix gives *eigenvalues* which can used to deduce algebraic connectivity of the graph. One way to look at this connection is the so-called node level expression of the Laplacian applied to the vertices,

$$L(v_i) = \sum_{v_i \sim v_j} (v_i - v_k). \tag{3}$$

Note that this is a local averaging formula and works in the same spirit as the diffusion operator Δ in the continuous domain.

For a graph, G, and its Laplacian matrix, L, with eigenvalues $\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{n-1}$: L is always positive semi-definite i.e. $\forall i, \lambda_i \geq 0, \lambda_0 = 0$. The number of connected components of G is given by the number of eigenvalues equal to zero. λ_0 is always zero. λ_1 gives the algebraic connectivity of the graph. Algebraic connectivity reflects how well-connected the graph is.

When the graph is a probabilistic roadmap, this information becomes even more revealing. The next section conducts spectral analysis on the roadmap by demonstrating the relationship between the roadmap and the spectrum of its Laplacian matrix. This finally relates to the discovery of narrow passages in the underlying space.

4. DETECTING NARROW PASSAGES

In this section, we give some empirical evidence of using spectral geometry for detecting narrow passages. We study a very simple two dimensional example shown in Figure 2. The robot lives in a space made up of two chambers connected by a narrow passage. It is also free to rotate about a pivot at its center. Thus the free configurations make up a bounded subset of $\mathbb{R}^2 \times \mathcal{S}^1$ [1, 2, 3].

At first, one might be misled into thinking that this is a *flat* space and the Laplacian operator Δ, may not be able to give good bounds on the spectral gap. Strictly speaking, this is only a metric space on which the Laplacian structure of a Riemannian manifold does not exactly hold. However, as we will soon see, it is enough to note for the sake of

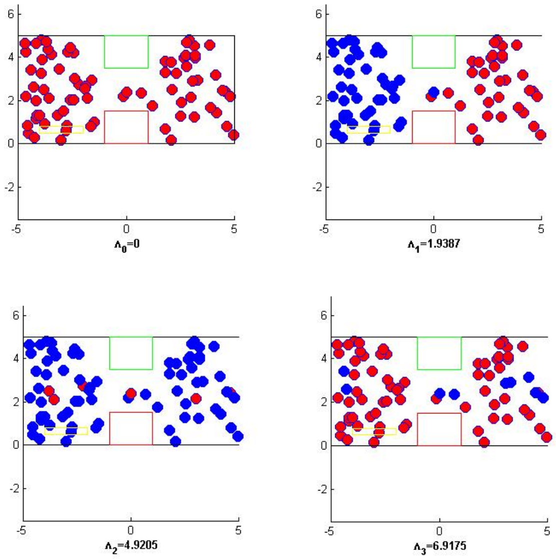

Figure 3: The first few eigenvectors of the PRM graph Laplacian.

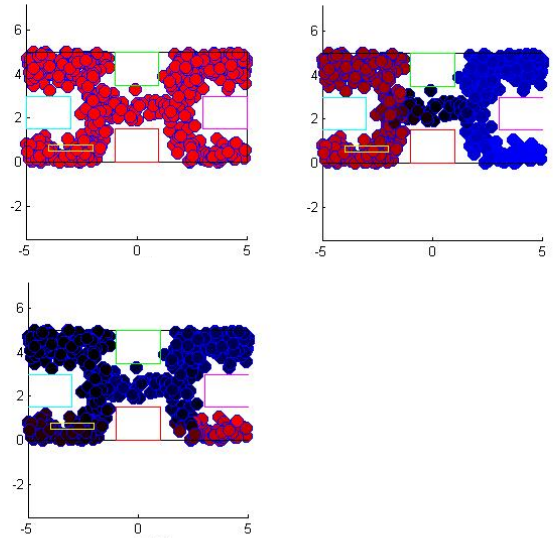

Figure 4: Another example of the first few eigenvectors of the PRM graph Laplacian.

getting an empirical evidence that the free space is actually of negative curvature.[1] Once this is established, inequalities 1 and 2 guide us on how the Laplacian spectrum looks like. We discover this negative curvature by two ways.

4.0.1 Gromov's link condition

First, assemble the space as a cubical complex by attaching squares of various sizes as suggested in the top right of Figure 2. Now apply *Gromov's link condition* at each vertex [24], which states that a cubical complex has non-positive curvature if and only if the link of each vertex is a flag complex. We note that for each node, the link is indeed a flag complex. One such link has been depicted in Figure 2. Therefore, the space has non-positive curvature.

4.0.2 Triangle comparison

A second way to see it is by considering the geodesics on this space. Mark three points on the space. Now connect them via the free space. Two such examples have been shown in the bottom of Figure 2. Note that if we draw comparison triangles of equal side-lengths, we would only be able to draw them in a space of negative curvature [24]. This verifies that the space has non-positive curvature. Therefore, we should expect to see estimates of λ_k driven by the inequalities 1 and 2.

Next we construct the PRM for this problem according to the guidelines of Section 2, and setup the milestones and edges as depicted in Figure 3. We compute the eigenvector decomposition of the PRM graph Laplacian and the results are as expected. Note that in the Figure, milestones have been colored red or blue according to the sign of the corresponding eigenvector component.

1. There is only one zero eigenvalue. This matches with the fact that the space has one connected component. The corresponding eigenvector is a constant (all red). In electrical engineer's terms, this is the DC value of the spectrum.

2. The next eigenvalue captures the spectral gap. Notice that it is significantly higher than zero (1.938 to be exact) thus revealing that the underlying space is negatively curved.

3. More importantly, the corresponding eigenvector changes sign right at the neck, thus capturing the notion of a narrow passage spectrally. This is the first harmonic of the space.

4. The next higher spectra do not reveal much information. This is also consistent since the space does not have any higher order features. In a space with multiple chambers and connecting narrow passages, these eigenvectors will also start to reveal more information about the space.

We repeated these experiments for various degrees of narrowness in the space and found the results to be consistent. The results of another experiment have been show in Figure 4.

5. CONCLUSIONS AND FUTURE WORK

In this paper, we have demonstrated a new connection between spectral geometry of configuration spaces and the notion of narrow passages in motion planning algorithms. The discrete structure of PRMs has been helpful in making this connection where we make use of spectral graph theory to reveal important information about the configuration space. We observe that these revelations are connected tightly with notions of curvature and geometry.

Thus far we have only demonstrated an empirical albeit very strong evidence of this new connection between spec-

[1]Work is underway to make this loose connection mathematically concrete.

tra and motion planning difficulties. Work in underway in making concrete analysis of these connections.

6. REFERENCES

[1] S. M. LaValle, *Planning Algorithms*, Cambridge University Press, 2006.

[2] Jean Claude Latombe, *Robot Motion Planning*, Kluwer Academic Publishers, Boston, MA, 1991.

[3] H. Choset, K.M. Lynch, S. Hutchinson, G. Kantor, W. Burgard, L.E. Kavraki, and S. Thrun, "Principles of Robot Motion : Theory, Algorithms, and Implementations," The MIT Press, 2005.

[4] L.E. Kavraki, P. Svestka, J.C. Latombe, and M.H. Overmars, "Probabilistic Roadmaps for Path Planning in High-Dimensional Configuration Space," *IEEE Trans. on Robotics & Automation*, 12(4):566Ű580, 1996.

[5] L.E. Kavraki, M.N. Kolountzakis, and J.C. Latombe, "Analysis of Probabilistic Roadmaps for Path Planning," *IEEE Trans. on Robotics & Automation*, 14(1):166Ű171, 1998.

[6] D. Hsu, J.C. Latombe, and R. Motwani, "Path Planning in Expansive Configuration Spaces," *In Proc. IEEE Int. Conf. on Robotics & Automation*, pages 2719Ű2726, 1997.

[7] D. Hsu, L.E. Kavraki, J.C. Latombe, R. Motwani, and S. Sorkin, "On finding Narrow Passages with Probabilistic Roadmap Planners," In P.K. Agarwal et al., editors, *Robotics: The Algorithmic Perspective: 1998 Workshop on the Algorithmic Foundations of Robotics*, pages 141Ű154. A. K. Peters, Wellesley, MA, 1998.

[8] D. Hsu, T. Jiang, J. Reif, and Z. Sun, "The Bridge Test for sampling Narrow Passages with Probabilistic Roadmap Planners," *In Proc. IEEE Int. Conf. on Robotics & Automation*, pages 4420Ű4426, 2003.

[9] D. Hsu, J.C. Latombe, and H. Kurniawati, "On the Probabilistic Foundations of Probabilistic Roadmap Planning," *Int. J. of Robotics Research*, 25(7):627-643, July 2006.

[10] Z. Sun, D. Hsu, T. Jiang, H. Kurniawati, and J. Reif, "Narrow Passage Sampling for Probabilistic Roadmap Planners," *IEEE Trans. on Robotics*, 21(6):1105Ű1115, 2005.

[11] H.H. González-Ba nos, D. Hsu, and J.C. Latombe, "Motion planning: Recent developments," In S.S. Ge and F.L. Lewis, editors, *Autonomous Mobile Robots: Sensing, Control, Decision-Making and Applications*, CRC Press, 2006.

[12] M. Saha, J.C. Latombe, Y.C. Chang, and F. Prinz, "Finding Narrow Passages with Probabilistic Roadmaps: The Small-step Retraction Method," *Autonomous Robots*, 19(3):301Ű319, 2005.

[13] R. Geraerts and M.H. Overmars, "A comparative study of Probabilistic Roadmap Planners," In J.D. Boissonnat et al., editors, *Algorithmic Foundations of Robotics V*, pages 43Ű59. Springer, 2002.

[14] V. Boor, M.H. Overmars, and F. van der Stappen, "The Gaussian Sampling strategy for Probabilistic Roadmap Planners," *In Proc. IEEE Int. Conf. on Robotics & Automation*, pages 1018Ű1023, 1999.

[15] J. Barraquand, L.E. Kavraki, J.C. Latombe, T.Y. Li, R. Motwani, and P. Raghavan, "A Random Sampling Scheme for Path Planning," *International Journal of Robotics Research*, 16(6):759-774, 1997.

[16] T. Simt'eon, J.P. Laumond, and C. Nissoux, "Visibility-based Probabilistic Roadmaps for Motion Planning," *J. Advanced Robotics*, 14(6):477Ű494, 2000.

[17] Daniel A. Spielman, "Spectral Graph Theory and its Applications," 48^{th} *Annual IEEE Symposium on Foundations of Computer Science*, FOCS 2007: Pages 29-38

[18] J. Jost, *Riemannian Geometry and Geometric Analysis*, (Universitext), Springer, 2008.

[19] I. Chavel, *Eigenvalues in Riemannian Geometry*, Vol 115, Pure & Applied Mathematics, 1984.

[20] M. Kac, "Can one hear the shape of a drum?", *American Mathematical Monthly*, vol. 73, no. 4, part 2, 1966.

[21] B. Eckmann, "Harmonische Funktionen und Randwertaufgaben in Einem Komplex," *Commentarii Math. Helvetici*, Vol. 17, 1945.

[22] M. Desbrun, A. Hirani, and J. Marsden, "Discrete Exterior Calculus for Variational Problems in Computer Graphics and Vision," in *Proc. 42nd IEEE Conference on Decision and Control*, 2003.

[23] R. Forman, "Combinatorial Differential Topology and Geometry," *New Perspectives in Geometric Combinatorics*, L. Billera et al. (eds.), Cambridge University Press, Math. Sci. Res. Inst. Publ. 38, 1999.

[24] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer, 2009.

[25] *http://parasol.tamu.edu/dsmft/benchmarks/mp/*