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Leader-Based Multi-Agent Coordination: Controllability and Optimal Control*

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Abstract— **In this paper, we consider the situation where a collection of leaders dictate the motion of the followers in heterogenous multi-agent applications. In particular, the followers move according to a decentralized averaging rule, while the leaders' motion is unconstrained. Thus, the trajectories of the leaders can be viewed as exogenous control inputs, which allows us to state and study questions concerning controllability and optimal control.**

I. Introduction

In the rapidly expanding field of multi-agent robotics, two distinctly different approaches have emerged, depending on whether any distinct agents are allowed to take on leader roles. This paper focus on the leader based control problem and we investigate a number of control theoretic issues that arise when the followers' dynamics are governed by consensus-like local interaction rules. This means that only the leaders are allowed to move freely, and we moreover assume that they have access to global information.

This paper is based on the results that have appeared during the last five years in the intersection of controls and algebraic graph theory. (For a representative sample, see [7], [11], [12], [14], [16], [17], [18], [19], [21], [22], [23], [25].) This body of work mainly deals with the leader-less or homogeneous situation. However, results have been obtained for heterogenous formations as well, including string-stability [24], leader-to-follower stability and control [5], [26], virtual leader-based control [6], [13], and formation control [3], [4]. In this paper we combine these two lines of thought by applying graph theoretic methods in the context of leader-based formation control.

Loosely speaking, one can think of the problem under investigation in this paper as the autonomous sheep-herding problem. In other words, how should the herding dogs move in order to maneuver the herd in the desired way? In particular, we study the issue of controllability, i.e. characterize conditions under which the leaders can move the followers to any desired position. Our main controllability result is a sufficient condition that depends both on the number of leaders as well as the network topology, which we assume to be static throughout the paper. This result is moreover constructive in that it allows us to select leaders for the purpose of rendering the system controllable.

Once a set of leaders is selected, we apply optimal control techniques for driving the system between specified positions. It is shown that this problem is in fact equivalent to the problem of driving an invertible linear system in a quasi-static equilibrium process[1] .

The outline of this paper is as follows: In Section 2, we present some basic enabling results. These are followed by a controllability study in Section 3, where a sufficient condition for controllability relates the relative homology of a particular graph to the homology of the original network. The point-to-point transfer problem is considered in Section 4, where it is shown that an optimal transfer is always possible regardless of controllably properties.

II. Problem Statement

We let the state of an individual agent be described by a vector in $\mathbb{R}^n$. Moreover, we assume that the dynamics along each dimension can be decoupled. Hence each dimension can be considered independently, and it is sufficient to analyze the performance along a single dimension. In other words, let $x_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, be the position vector of the i th agent, and let $x = [x_1, x_2, \ldots, x_N]^T$ be the state vector of the group of agents, where N is the total number of agents. For such a system, a widely adopted distributed control strategy for driving the agents to a common point (the rendezvous problem) is given by [11]

$$\dot{x}_i = \sum_{j \in N(i)} (x_j - x_i), \qquad (1)$$

where $N(j)$ encodes the fact that the information is allowed to flow from agent j to agent i.

Graph theory can provide a variety of tools for analyzing such control strategies [8]. A graph $\mathcal{G} = (V, E)$ consists of a set of nodes $V = \{v_1, \ldots, v_N\}$, which corresponds to the different agents, and a set of edges $E \subset V \times V$, which relates to a set of unordered pairs of agents. Note that $(v_i, v_j) = (v_j, v_i) \in E$, if and only if a communication link exists between agents i and j, and we will use $V(\mathcal{G})$ and $E(\mathcal{G})$ to denote the node and edge sets respectively. Graph

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[1]A process is called quasi-static when it slowly transfers from one equilibrium state to another such that the system is always close to a equilibrium state.

theory moreover tells us that the control law in (1) can be written as

$$\dot{x} = -\mathcal{L}(\mathcal{G})x, \tag{2}$$

where $\mathcal{L}(\mathcal{G})$ is the graph Laplacian for $\mathcal{G}$. For the definition of $\mathcal{L}(\mathcal{G})$ and related properties, please refer to [8].

Now, in some applications, we can easily imagine a subset of the agents as having superior sensing and communication abilities. We denote this subset as the leader set, and the remaining agents make up the follower set. As a result x can be divided into two parts, the states of the leaders x_l and those of the followers x_f. Here the subscript f and l denote $followers$ and $leaders$ respectively. If we reorganize the indexing of the agents in such a way that the first N_f agents are followers and the rest N_l of them are leaders, where $N_f + N_l = N$, the Laplacian matrix can be partitioned as

$$\mathcal{L}(\mathcal{G}) = \begin{bmatrix} \mathcal{L}_f & l_{fl} \\ l_{fl}^T & \mathcal{L}_l \end{bmatrix}, \tag{3}$$

where $\mathcal{L}_f \in \mathbb{R}^{N_f \times N_f}$, $\mathcal{L}_l \in \mathbb{R}^{N_l \times N_l}$ and $l_{fl} \in \mathbb{R}^{N_f \times N_l}$. A useful fact about the Laplacian is that it can be obtained from the product of incidence matrices,

$$\mathcal{L}(\mathcal{G}) = \mathcal{I}(\mathcal{G}^\sigma)\mathcal{I}(\mathcal{G}^\sigma)^T,$$

where σ is an arbitrary orientation assignment to the edges of the graph, $\mathcal{I} \in \mathbb{R}^{N \times M}$ is the incidence matrix, $N = |V(\mathcal{G})|$ and $M = |E(\mathcal{G})|$. By writing the incidence matrix as $\mathcal{I}^T = [\mathcal{I}_f^T, \mathcal{I}_l^T]^T$, where $\mathcal{I}_f \in \mathbb{R}^{N_f \times M}$ and $\mathcal{I}_l \in \mathbb{R}^{N_l \times M}$ (note here we have dropped the explicit dependence on σ), we get

$$\mathcal{L}_f = \mathcal{I}_f \mathcal{I}_f^T, \ \mathcal{L}_l = \mathcal{I}_l \mathcal{I}_l^T \text{ and } l_{fl} = \mathcal{I}_f \mathcal{I}_l^T. \tag{4}$$

Lemma 2.1: If $\mathcal{G}$ is connected, then $\mathcal{L}_f$ is positive definite,

Proof: It is well known that $\mathcal{L}(\mathcal{G}) \succeq 0$. In addition, if $\mathcal{G}$ is connected, we have that $\mathcal{N}(\mathcal{L}(\mathcal{G})) = \text{span}\{\mathbf{1}\}$, where $\mathcal{N}(\cdot)$ denotes the null space and $\mathbf{1}$ is the vector with all entries being one. Now, since

$$x_f^T \mathcal{L}_f x_f = [x_f^T \ 0]\mathcal{L} \begin{bmatrix} x_f \\ 0 \end{bmatrix}$$

and $[x_f^T \ 0]^T \notin \mathcal{N}(\mathcal{G})$, we have that

$$[x_f^T \ 0]\mathcal{L} \begin{bmatrix} x_f \\ 0 \end{bmatrix} > 0 \ \forall \ x_f \in \mathbb{R}^{N_f},$$

and the lemma follows [2] ■

The rendezvous control law in (2) works because it averages the contribution from all neighbors. As such, it seems like a natural starting point when determining the movement of the followers. In other words, we will choose to let

$$\dot{x}_f = -\mathcal{L}_f x_f - l_{fl} x_l, \tag{5}$$

which allows us to state the following theorem.

Theorem 2.2: Given fixed leader positions x_l, the equilibrium point under the follower dynamics in (5) is

$$x_f = -\mathcal{L}_f^{-1} l_{fl} x_l, \tag{6}$$

[2]Note that an alternative yet significantly more involved version of this proof can be found in [2].

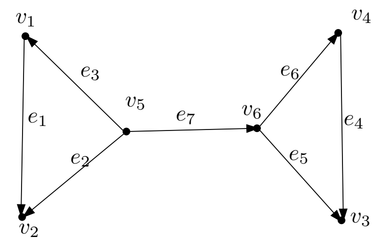

Fig. 1. An example graph used for choosing leaders such that the condition in Theorem 3.2 is satisfied.

which is globally asymptotically stable.

Proof: From the previous lemma, we know that $\mathcal{L}_f$ is invertible and hence (6) is well-defined. Hence the equilibrium point is unique. Moreover, since $\mathcal{L}_f \succ 0$, this equilibrium point is in fact globally asymptotically stable. ■

III. Controllability analysis of the leader-follower scheme

In order for the followers to achieve their desired positions in finite time, we will employ optimal control techniques. But, before we do so, the issue of controllability should be settled. In this section, we will thus discuss the controllability issue and provide a sufficient condition, starting from the following well-known result.

Proposition 3.1: Given $x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, the system $\dot{x} = Ax + Bu$ is controllable if and only if, $\forall \ v_i \in \text{spec}(A)$, where spec$(\cdot)$ denotes the spectrum, $v_i \notin \mathcal{N}(B^T)$.
The proof of this can be found in many control theory textbooks (e.g. [1]) and will be omitted here.

Theorem 3.2: The system $(\mathcal{L}_f, l_{fl})$ is controllable if $\mathcal{G}$ is connected and $\mathcal{N}(\mathcal{I}_l) \subseteq \mathcal{N}(\mathcal{I}_f)$.

Proof: For the system $(\mathcal{L}_f, l_{fl})$, the condition in Proposition 3.1 translates to $v_i(\mathcal{L}_f) \notin \mathcal{N}(l_{fl}^T), \ \forall \ v_i \in \text{spec}(\mathcal{L}_f)$, or $\mathcal{I}_l \mathcal{I}_f^T v_i \neq 0 \ \forall v_i \in \text{spec}(\mathcal{L}_f)$. Thus if $\mathcal{N}(\mathcal{I}_l) \subseteq Im(\mathcal{I}_f^T)^\perp = \mathcal{N}(\mathcal{I}_f)$, the system is controllable. ■

Note that as a consequence of Theorem 3.2, we have a constructive way of assigning leadership roles to agents in order to ensure controllability.

Given a network topology, first find the null space of $\mathcal{I}$. Then select the appropriate rows of $\mathcal{I}$ and stack them in a new matrix such that the null space of this new matrix is contained in $\mathcal{N}(\mathcal{I})$. As an example, consider the directed graph in Figure 1, where we have

$$\mathcal{I} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}$$

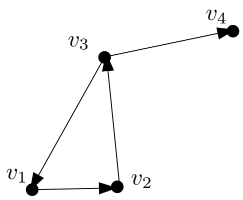

Fig. 2. The original graph corresponding to the homology $H(\mathcal{G})$.

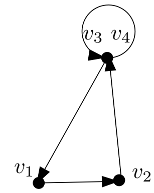

Fig. 3. The quotient graph corresponding to the relative homology $H(\mathcal{G}/\mathcal{F})$.

with

$$\mathcal{N}(\mathcal{I}) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

From the incidence matrix we directly see that by choosing any single agent as follower and the remaining five as leaders, Theorem 3.2 will be satisfied.

It is worth noticing that this sufficient condition is conservative. In many cases we can find configurations with less leaders, which is still controllable. For instance, in Figure 1, we can choose nodes v_1, v_2, v_3 and v_4 as leaders and the system is still controllable.

Now, algebraic topology provides us with additional tools for shedding light on the result of Theorem 3.2. In fact, the incidence matrix $\mathcal{I}$ is a matrix representation of the boundary operator

$$\partial : C_1(\mathcal{G}) \to C_0(\mathcal{G}),$$

where $C_i(\mathcal{G})$ is the vector space (chain complex) whose basis consists of all i-simplices in $\mathcal{G}$ (the directed graph)[3].

As an example, let $\mathcal{G}$ be given by the graph in Figure 2. The 0-simplices are $[v_1], [v_2], [v_3], [v_4]$, while the 1-simplices are $[v_1, v_2], [v_2, v_3], [v_3, v_1], [v_3, v_4]$.

Now, $Ker(\partial) = \{S \in C_1(\mathcal{G}) \mid \partial S = 0\}$ is equal to the first homology $H_1(\mathcal{G})$, i.e. the set of cycles. Technically, it is the set of cycles that are not themselves "boundaries" to higher simplices, but no such simplices are present in this paper. If we return to the example in Figure 2, $Ker(\partial) = \text{span}\{[v_1, v_2] + [v_2, v_3] + [v_3, v_1]\}$, since

$$\begin{aligned} &\partial([v_1, v_2] + [v_2, v_3] + [v_3, v_1]) \\ &= [v_2] - [v_1] + [v_3] - [v_2] + [v_1] - [v_3] \\ &= 0. \end{aligned}$$

Moreover, if we equate e_1 with $[v_1, v_2]$, e_2 with $[v_2, v_3]$, e_3 with $[v_3, v_1]$, and e_4 with $[v_3, v_4]$, the incidence matrix is

$$\mathcal{I} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

[3]Throughout this part we will assume that the field in question is $\mathbb{R}$, i.e. write $C_i(\mathcal{G})$ instead of $C_i(\mathcal{G}; \mathbb{R})$.

and

$$\mathcal{N}(\mathcal{I}) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\},$$

i.e. the same cycle as in $Ker(\partial)$ makes up the null-space, as should be expected.

Now consider the graph obtained by equating all followers with a single vertex. In other words, it is the "quotient graph" $\mathcal{G}/\mathcal{F}$, where $\mathcal{F}$ is the subgraph containing all follower vertices but no edges. We can then define the relative "chain complex" $C_1(\mathcal{G}/\mathcal{F}) = C_1(\mathcal{G})/C_2(\mathcal{F})$, and the corresponding boundary operator

$$\partial : C_1(\mathcal{G}/\mathcal{F}) \to C_0(\mathcal{G}/\mathcal{F}),$$

whose kernel (the relative cycles) encodes the null-space of $\mathcal{I}_l$, which gives us the following as a direct reformulation of Theorem 3.2.

Corollary 3.3: The system $(\mathcal{L}_f, l_{fl})$ is controllable if $\mathcal{G}$ is connected and $H_1(\mathcal{G}) = H_1(\mathcal{G}/\mathcal{F})$.

As an illustration, let v_1 and v_2 be leaders in Figure 2, which gives us $\mathcal{G}/\mathcal{F}$ as shown in Figure 3. Since the first relative homology group consists of two cycles while the original system had only one cycle, we know that this is not a controllable system. This can moreover be seen from

$$\mathcal{I}_l = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix},$$

whose null-space is

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

where $[1\ 1\ 1\ 0]^T$ corresponds to the original cycle in $\mathcal{G}$, while $[0\ 0\ 0\ 1]^T$ corresponds to the new cycle in $\mathcal{G}/\mathcal{F}$. What this moreover means is that in order for the sufficient controllability condition to be satisfied, no followers can have edges in-between them.

Now that a leader-follower structure has been chosen and the controllability issue has been studied, the next problem is how to move the system from one equilibrium to the next in finite time. The following section will give an optimal control answer to this problem.

IV. Optimal Control of Quasi-Equilibrium Processes

For the sake of notational convenience, we will equate x_f with x, and x_l with u throughout this section. Moreover we will identify A with $-\mathcal{L}_f$ and B with $-l_{fl}$.

Using this notation the system in (5) can be rewritten as

$$\dot{x} = Ax + Bu. \tag{7}$$

Moreover, since the leaders are unconstrained in their motion, we let

$$\dot{u} = v,$$

where v is the control input.

For a fixed u, the quasi-static equilibrium to (7) is given by

$$x = -A^{-1}Bu. \tag{8}$$

The problem under consideration here is the quasi-static equilibrium process problem, i.e. the problem of transferring (x, u) from an initial point satisfying (8) to a final point also satisfying (8). We moreover want to achieve this in a finite amount of time, and we define our performance function as follows

$$J = \frac{1}{2}\int_0^T (\dot{x}^T P\dot{x} + \dot{u}^T Q\dot{u})dt, \tag{9}$$

where $P \succeq 0$ and $Q \succ 0$. The optimal control problem can be formulated as

$$\min_v J. \tag{10}$$

Since the previous section dealt with the issue of controllability, it would be tempting to assume that the leader-follower system must be controllable. However, no such assumptions are need, even though it is well-known that the general point to point transfer problem has a solution if (A, B) is controllable. In fact, in this case we can remove this condition completely. To see this assume, without loss of generality, that we have a standard controllable decomposition as

$$\dot{x} = \begin{bmatrix} \dot{x}_c \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u,$$

where x_c is the controllable part and x_u is the uncontrollable part. Now, given a fixed u^e, where the superscript e denotes equilibrium, the quasi-static equilibrium is given by

$$0 = \begin{bmatrix} A_{11}x_c^e + A_{12}x_u^e + B_1u^e \\ A_{22}x_u^e \end{bmatrix}.$$

Since A is invertible (and hence also A_{22}), this means that $x_u^e = 0$. Hence the quasi-static process will simply drive $x_u(0) = 0$ to $x_u(T) = 0$ and we can restrict our attention to the non-trivial part of the system

$$\dot{x}_c = A_{11}x_c + A_{12}x_u + B_1u.$$

But, since $x_u(t) = 0$ on the interval $[0, T]$, we only have

$$\dot{x}_c = A_{11}x_c + B_1u,$$

and since (A_{11}, B_1) is a controllable pair, the point-to-point transfer is always possible.

Now in order to solve this problem, we first form the Hamiltonian

$$\begin{aligned} H =& \frac{1}{2}(\dot{x}^T P\dot{x} + \dot{u}^T Q\dot{u}) + \lambda^T(Ax + Bu) + \mu^T v \\ =& \frac{1}{2}(x^T A^T PAx + 2x^T A^T PBu + u^T B^T PBu + v^T Qv) \\ &+ \lambda^T(Ax + Bu) + \mu^T v, \end{aligned} \tag{11}$$

where λ and μ are the co-states. The first order necessary optimality condition then gives

$$\begin{aligned} \frac{\partial H}{\partial v} &= v^T Q + \mu^T = 0 \quad \Rightarrow v = -Q^{-1}\mu, \\ \dot{\lambda} &= -(\frac{\partial H}{\partial x})^T = -A^T PAx - A^T PBu - A^T\lambda, \\ \dot{\mu} &= -(\frac{\partial H}{\partial u})^T = -B^T PAx - B^T PBu - B^T\lambda. \end{aligned} \tag{12}$$

In other words, by letting $z = [x^T, u^T, \lambda^T, \mu^T]^T$, we obtain the following equation

$$\dot{z} = Mz, \tag{13}$$

where

$$M = \begin{bmatrix} A & B & 0 & 0 \\ 0 & 0 & 0 & -Q^{-1} \\ -A^T PA & -A^T PB & -A^T & 0 \\ -B^T PA & -B^T PB & -B^T & 0 \end{bmatrix}.$$

Let the initial state be given by

$$z_0 = [x_0^T, u_0^T, \lambda_0^T, \mu_0^T]^T.$$

Now, the problem is to select λ_0 and μ_0 in such a way that, through this choice, we get

$$\begin{aligned} u(T) &= u_T \\ x(T) &= -A^{-1}Bu_T \triangleq x_T. \end{aligned}$$

In order to achieve this, we partition the matrix exponential in the following way

$$e^{MT} = \begin{bmatrix} \phi_{xx} & \phi_{xu} & \phi_{x\lambda} & \phi_{x\mu} \\ \phi_{ux} & \phi_{uu} & \phi_{u\lambda} & \phi_{u\mu} \\ \phi_{\lambda x} & \phi_{\lambda u} & \phi_{\lambda\lambda} & \phi_{\lambda\mu} \\ \phi_{\mu x} & \phi_{\mu u} & \phi_{\mu\lambda} & \phi_{\mu\mu} \end{bmatrix}. \tag{14}$$

We can find the initial conditions of the co-states by solving

$$\begin{bmatrix} x_T \\ u_T \end{bmatrix} = \begin{bmatrix} \phi_{xx} & \phi_{xu} & \phi_{x\lambda} & \phi_{x\mu} \\ \phi_{ux} & \phi_{uu} & \phi_{u\lambda} & \phi_{u\mu} \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \\ \lambda_0 \\ \mu_0 \end{bmatrix}.$$

Now, let

$$\Phi_1 = \begin{bmatrix} \phi_{xx} & \phi_{xu} \\ \phi_{ux} & \phi_{uu} \end{bmatrix} \text{ and } \Phi_2 = \begin{bmatrix} \phi_{x\lambda} & \phi_{x\mu} \\ \phi_{u\lambda} & \phi_{u\mu} \end{bmatrix},$$

which gives

$$\begin{bmatrix} \lambda_0 \\ \mu_0 \end{bmatrix} = \Phi_2^{-1}\left(\begin{bmatrix} x_T \\ u_T \end{bmatrix} - \Phi_1 \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right).$$

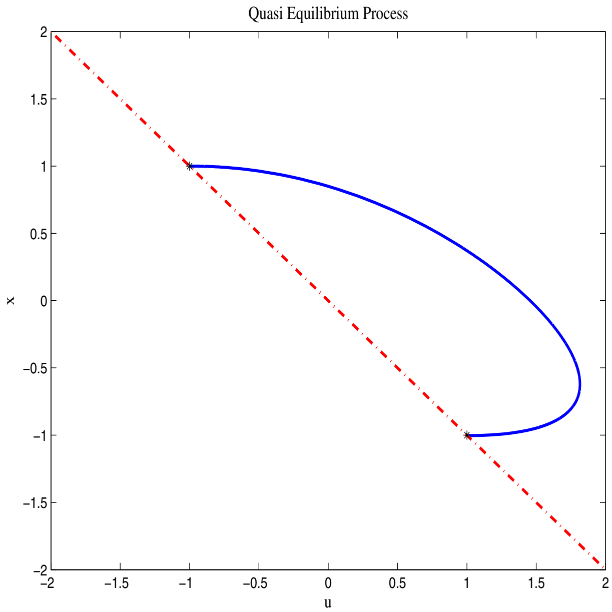

Fig. 4. An example of quasi-static process for the system $\dot{x} = -x - u$, $P = Q = 1, x_0 = 1, u_0 = -1, x_T = -1, u_T = 1$.

Since we are considering a quasi-static process, we have

$$x_0 = -A^{-1}Bu_0 \text{ and } x_T = -A^{-1}Bu_T,$$

and hence the initial conditions of the co-states become

$$\begin{bmatrix} \lambda_0 \\ \mu_0 \end{bmatrix} = -\Phi_2^{-1}\Psi \begin{bmatrix} u_0 \\ u_T \end{bmatrix},$$

where

$$\Psi = \begin{bmatrix} \phi_{xx}A^{-1}B - \phi_{xu} & -A^{-1}B \\ \phi_{ux}A^{-1}B - \phi_{uu} & I \end{bmatrix}.$$

The invertibility of Φ_2 follows directly from the fact that this particular point-to-point transfer problem always has a unique solution, as observed previously.

As an example, Figure 4 shows a quasi-static process, where the dynamics of the system is given by

$$\dot{x} = -x - u,$$

and P and Q are both set to be 1. The system starts from $x_0 = 1, u_0 = -1$ and the desired final position is $x_T = -1, u_T = 1$. The dash-dotted line shows the subspace $\{(x, u) \mid x = -A^{-1}Bu\}$, while the solid line is the actual trajectory of the system under the optimal control law with $T = 2$.

In Figure 5, snap shots of a herding process are shown where the leaders (black) move the followers (white) from an initial position to a final position. The leaders' initial and final positions are $x_{l0} = \{(-1, -1), (0, 1), (1, -1)\}$ and $x_{lT} = \{(-1, -1), (0, 1), (1, -1)\}$ respectively. The followers' position (equilibria) are determined by (6) and they follows the control law (5). The time horizon is set to be one second. P and Q are identity matrices with appropriate dimensions.

One may notice that there are instances when the follower is not inside the convex hull spanned by their leaders. This happens because the time horizon is finite.

V. Conclusion

In this paper we study the leader-follower problem when the followers' dynamics are given by decentralized averaging (consensus-like) rules. The leaders' positions are taken to be the system inputs. A sufficient condition for controllability is given together with optimal centralized algorithms for driving this system from one quasi-static equilibrium point to another.

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References

[1] W. L. Brogan, Modern Control Theory, Prentice-Hall, Englewood Cliffs, NJ, 3rd Edition, 1990.

[2] D. Chen and S. Toledo, "Combinatorial Characterization of the Null Spaces of Symmetric H-Matrices," *Linear Algebra and its Applications*, 392: pp.71-90, 2004.

[3] T. Eren, W. Whiteley, B.D.O. Anderson, A.S. Morse, P.N. Belhumeur, "Information structures to secure control of rigid formations with leader-follower architecture" *Proceedings of the American Control Conference, 2005* , pp. 2966-2971, June 8-10, 2005.

[4] Jonathan Lawton, Randal Beard, Brett Young, "A Decentralized Approach To Formation Maneuvers," *IEEE Transactions on Robotics and Automation*, Vol. 19, No. 6, pp. 933-941, Dec. 2003.

[5] J. Desai, J. P. Ostrowski and V. Kumar, "Controlling formations of multiple mobile robots," *Proc. IEEE Int. Conf. Robot. Automat.*, Leuven, Belgium, pp.2864-2869, May 1998.

[6] M. Egerstedt and X. Hu, "Formation Constrained Multi-Agent Control," *IEEE Transactions on Robotics and Automation*, Vol. 17, No. 6, pp. 947-951, Dec. 2001.

[7] J.A. Fax and R.M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Transactions on Automatic Control*, Vol. 49 , Issue: 9 , Sept. 2004 pp. 1465 - 1476.

[8] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer, 2001.

[9] A. Hatcher, *Algebraic Topology*, Cambridge, 2002.

[10] H. Axelsson, M. Abubakr, and M. Egerstedt, "Autonomous Formation Switching for Multiple, Mobile Robots," *IFAC Conference on Analysis and Design of Hybrid Systems*, Saint-Malo, Brittany, France, June 2003.

[11] A. Jadbabaie, J. Lin and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest rules," *IEEE Transaction on Automatic Control*, Vol. 48, No. 6, pp.988-1001, 2003.

[12] M. Ji and M. Egerstedt, "Connectedness Preserving Distributed Coordination Control over Dynamic Graphs," *Proceedings of the American Control Conference, 2005*, pp. 93-98, Portland OR, June 8-10, 2005.

[13] N. Leonard and E. Friorelli, "Virtual leaders, artificial potentials and coordinated control of groups," *Proc. of the 40th IEEE Conference on Decision and Control 2001*, Vol. 3, pp. 2968-2973, 4-7 Dec. 2001.

[14] Z. Lin, M. Broucke, and B. Francis, "Local control strategies for groups of mobile autonomous agents," *IEEE Trans. Automat. Contr.*, Vol. 49, No. 4, pp. 622-629, 2004.

[15] J. Lin, A.S. Morse, and B.D.O. Anderson, "The Multi-Agent Rendezvous Problem," *Proceedings of the 42nd IEEE Conference on Decision and Control*, pp.1508-1513, Maui, Hawaii USA, Dec. 2003.

[16] S. Martìnez, F. Bullo, J. Cortés, and E. Frazzoli. "On Synchronous Robotic Networks - Part I: Models, Tasks and Complexity Notions", *IEEE Transactions on Automatic Control*, April 2005. Note: Submitted.

[17] S. Martìnez, F. Bullo, J. Cortés, and E. Frazzoli. "On Synchronous Robotic Networks - Part II: Time Complexity of Rendezvous and Deployment Algorithms", *IEEE Transactions on Automatic Control*, April 2005. Submitted.

[18] M. Mesbahi, "On a dynamic extension of the theory of graphs," *Proceedings of the 2002 American Control Conference*, Vol. 2, pp. 1234 - 1239 , 8-10 May 2002.

[19] M. Mesbahi. "State-dependent graphs," *Proceedings of the 42nd IEEE Conference on Decision and Control*, pp. 3058-3063, Maui, Hawaii USA, Dec. 2003.

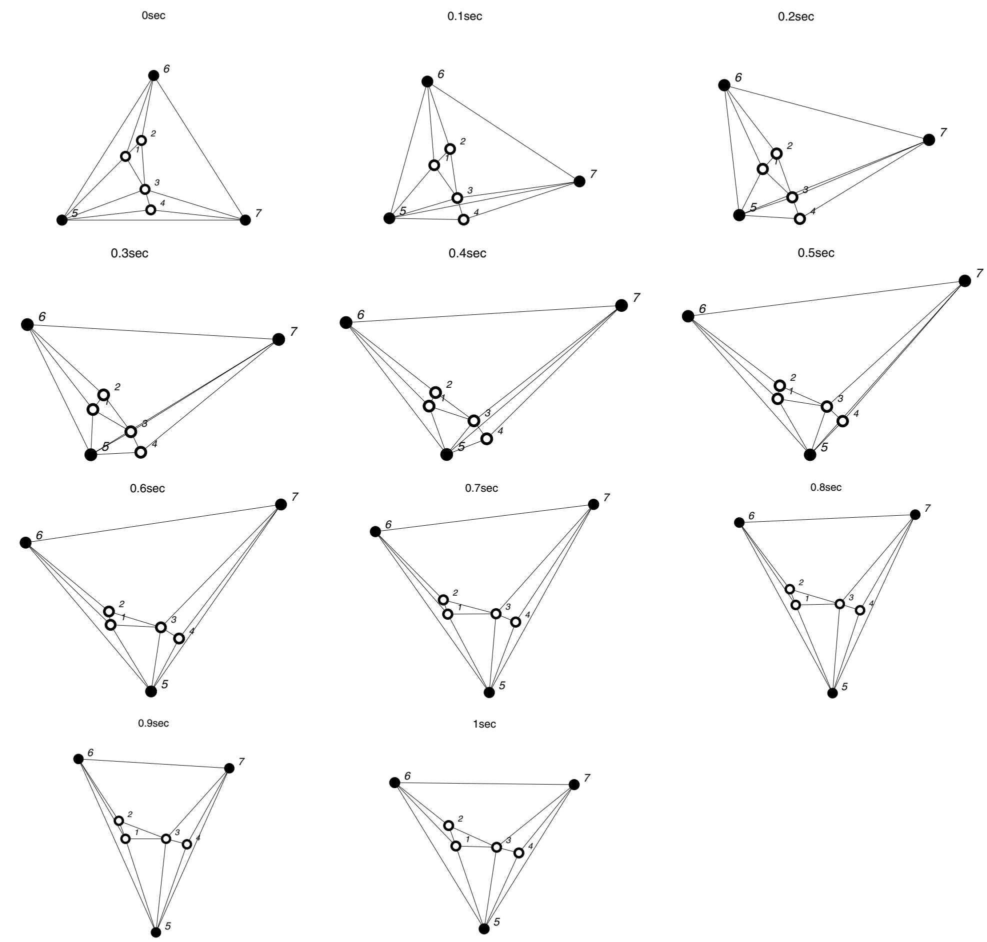

Fig. 5. A quasi-static process where 3 leaders (black nodes) herd 4 followers (white nodes), where $T = 1\ sec$.

[20] W. Ren, R.W. Beard, "Consensus of information under dynamically changing interaction topologies," *Proceedings of the American Control Conference, 2004*, Vol. 6, pp. 4939 - 4944, June 30-July 2, 2004.

[21] R. O. Saber, R. M. Murray, "Flocking with obstacle avoidance: cooperation with limited communication in mobile networks," *Proceedings of the 42nd IEEE Conference on Decision and Control*, pp. 2022-2028, Maui, Hawaii USA, Dec. 2003.

[22] R.O. Saber and R.M. Murray, "Agreement Problems in Networks with Directed Graphs and Switching Toplogy," *Proceedings of the 42nd IEEE Conference on Decision and Control, 2003*, Vol. 4, pp. 4126-4132, 9-12 Dec. 2003 .

[23] R.O. Saber and R.M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, Vol. 49, pp. 1520-1533, Issue: 9, Sept. 2004 .

[24] D. Swaroop, J.K. Hedrick, "String Stability of Interconnected Systems," *IEEE Transactions on Automatic Control*, Vol. 41, Issue 3, pp. 349-357, March 1996.

[25] H. Tanner, A. Jadbabaie, and G. J. Pappas, "Flocking in fixed and switching networks," *IEEE Transactions on Automatic Control*, Submitted April, 2005.

[26] Herbert G. Tanner, George J. Pappas and Vijay Kumar, "Leader-to-Formation Stability", *IEEE Transactions on Robotics and Automation*, Vol. 20, No. 3, pp. 433-455, June 2004.