Spectral Properties of Expansive Configuration Spaces: An Empirical Study

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Outline

- Review of Probabilistic roadmap algorithm
- Spectral geometry of spaces
- Spectra of PRM graphs
- Connections with CSpace geometry
- Simulation results
- Conclusions
Probabilistic Roadmaps

- Motion planning in complex configuration spaces
- Avoid difficulties in explicit geometric representations
- **One solution**: Sampling based methods

Probabilistic Roadmaps

Steps
1. Preprocessing phase
2. Query phase

BUILD_ROADMAP
1. $\mathcal{G}.\text{init}(); i \leftarrow 0$;
2. while $i < N$
3. if $\alpha(i) \in C_{\text{free}}$ then
4. $\mathcal{G}.\text{add}\_\text{vertex}(\alpha(i)); i \leftarrow i + 1$;
5. for each $q \in \text{NEIGHBORHOOD}(\alpha(i), \mathcal{G})$
6. if ((not $\mathcal{G}.\text{same}\_\text{component}(\alpha(i), q)$) and $\text{CONNECT}(\alpha(i), q)$) then
7. $\mathcal{G}.\text{add}\_\text{edge}(\alpha(i), q)$;

Ref. Planning Algorithms by Steve LaValle, 2006

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Probabilistic Roadmaps

• How much sampling is needed for a good representation?
• Expansiveness parameterized by $(\alpha, \beta, \epsilon)$

$$LOOKOUT(G) = \{q \in G \mid \mu(V(q) \setminus G) \geq \beta \mu(F' \setminus G)\}$$

$$\mu(LOOKOUT(G)) \geq \alpha \mu(G)$$

- Failure Probability

$$Pr < \frac{c_1}{\varepsilon \alpha} \exp\left(c_2 \varepsilon \alpha \left(-N + \frac{c_3}{\beta}\right)\right)$$

- Question: Can you estimate $(\alpha, \beta, \epsilon)$?


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Kac (1966) posed the problem … “Can you hear the shape of a drum?”

Laplacian operator on manifolds

\[ \Delta f = \frac{1}{\sqrt{g}} \sum_{j,k} \partial_j \left( g^{j,k} \sqrt{g} \partial_k f \right) \]

e.g. \[ \Delta f = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f \]

Nuemann / Dirichlet problem

\[ \Delta \phi = -\lambda \phi \]

\[ 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \]
Spectral Geometry of Spaces

What is known about the spectra?

- Weyl’s formula
  \[ N(\lambda) \sim \text{vol}(M)\text{vol}(B_1^n) \frac{\lambda^{n/2}}{(2\pi^n)} \]

- Rayleigh conjecture
  \[ \lambda_1(\Omega) \leq \lambda_1(B_1^n) \]

- Payne-Polya-Weinberger
  \[ \frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(B_1^n)}{\lambda_1(B_1^n)} \]

- Curvature bound
  \[ \lambda_1(\Omega) \geq -(n - 1)^2 \kappa/4 \]

- Cheeger’s inequality
  \[ \lambda_1(M) \geq \frac{h^2(M)}{4} \]

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Spectra of PRM graphs

- Adjacency matrix $A$
- Degree Matrix $D$
- Laplacian Matrix $L = D - A$

\[
\ell_{i,j} := \begin{cases} 
\text{deg}(v_i) & \text{if } i = j \\
-1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\
0 & \text{otherwise.}
\end{cases}
\]

- A discrete version of Laplacian operator
- Spectrum calculation is simple linear algebra

\[
L \psi_i = \lambda_i \psi_i
\]

\[
\lambda_1(L) = \inf_{\omega \in \mathbb{R}^n, \omega^T \omega = 1} \left( \frac{\omega^T L \omega}{\omega^T \omega} \right)
\]

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Controlled Experiments
Simulations

Obs 1: Size of Narrow Passage $\sim 2^{\text{nd}}$ Lowest Eigenvalue

Expansiveness (standard def)

Spectral expansiveness ($2^{\text{nd}}$ Lowest Eigenvalue $= 0.21$).

Plot of $2^{\text{nd}}$ eigenvector: Red (+ve), Blue (-ve), Color change at neck.

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## Simulations

### Spectral Expansiveness

<table>
<thead>
<tr>
<th>Size of Obstruction</th>
<th>2(^\text{nd} ) Eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Obstruction</td>
<td>1</td>
</tr>
<tr>
<td>Small</td>
<td>0.21</td>
</tr>
<tr>
<td>Bigger</td>
<td>0.0865</td>
</tr>
<tr>
<td>Biggest</td>
<td>0.013158</td>
</tr>
<tr>
<td>Almost two parts</td>
<td>0.000358</td>
</tr>
<tr>
<td>Fully Obstructed</td>
<td>0</td>
</tr>
</tbody>
</table>

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Simulations

Obs 2: Number of Narrow Passages ~ Spectral Jump

<table>
<thead>
<tr>
<th>Eigenvalues</th>
<th>Single Obstruction</th>
<th>2 Obstruction</th>
<th>4 Obstructions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt;</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt;</td>
<td>21.8647</td>
<td>5.4205</td>
<td>1.6533</td>
</tr>
<tr>
<td>3&lt;sup&gt;rd&lt;/sup&gt;</td>
<td>169.036</td>
<td>15.1546</td>
<td>4.5350</td>
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<tr>
<td>4&lt;sup&gt;th&lt;/sup&gt;</td>
<td>171.785</td>
<td>113.7700</td>
<td>7.0097</td>
</tr>
<tr>
<td>5&lt;sup&gt;th&lt;/sup&gt;</td>
<td>172.0000</td>
<td>114.4230</td>
<td>9.2204</td>
</tr>
<tr>
<td>6&lt;sup&gt;th&lt;/sup&gt;</td>
<td>174.701</td>
<td>119.0040</td>
<td>37.0506</td>
</tr>
</tbody>
</table>

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Simulations

Obs 2: Number of Narrow Passages $\sim$ Spectral Jump

One Obstructions

Two Obstructions

Three Obstructions

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Configuration Labeling

- Node Visibility
  ~ 1/eigenvector scale

- Detect narrow passages by appropriate threshold

- Visualization method for high dimensional CSpace
A concrete relationship?


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Conclusions

• Defined new spectral metrics to compare motion planning difficulties in configuration spaces
  – Computable with reasonable ease (Linear Algebra)
  – Alternative characterization (Frequency Domain)

• Detection of Narrow passages

• Labeling of “difficult” and “easy” parts of the configuration space
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