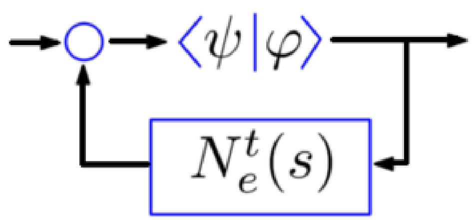
Spectral Properties of Expansive Configuration Spaces: An Empirical Study



Mhequb Hayat and **Abubakr Muhammad**

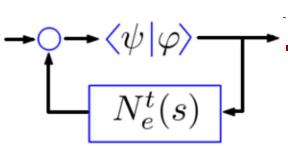
Laboratory for Cyber Physical Networks and Systems (CYPHYNETS)
LUMS School of Science & Engineering (SSE)
Lahore, Pakistan

May 12, 2011

IEEE International Conference on Robotics and Automation (ICRA 2011), Shanghai

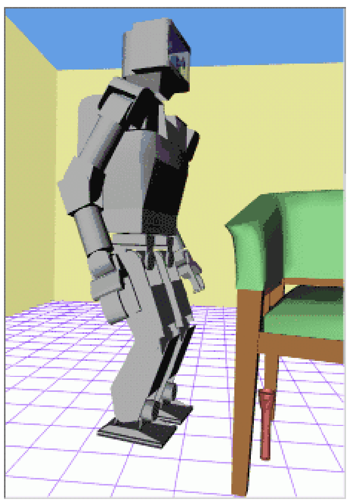
Outline

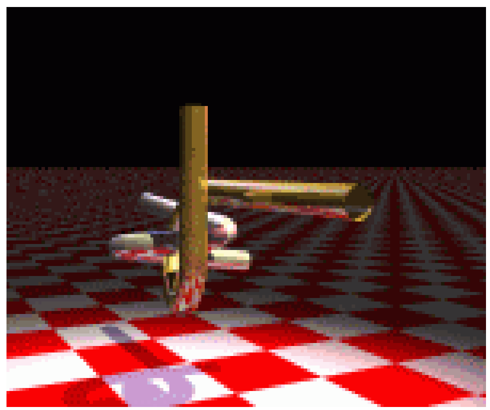
- Review of Probabilistic roadmap algorithm
- Spectral geometry of spaces
- Spectra of PRM graphs
- Connections with CSpace geometry
- Simulation results
- Conclusions

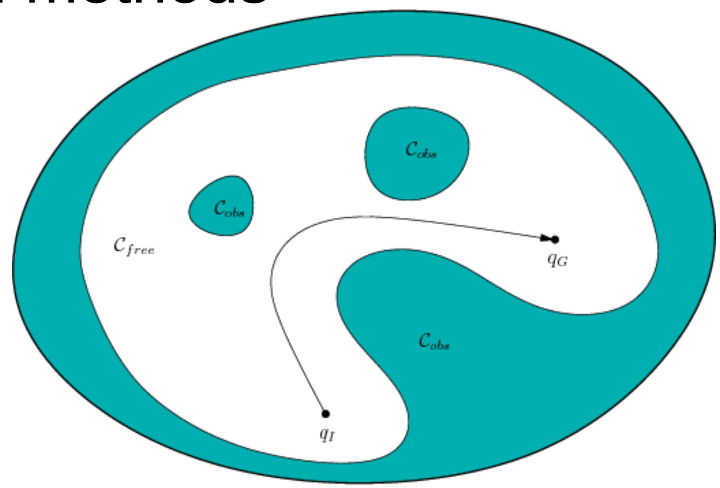


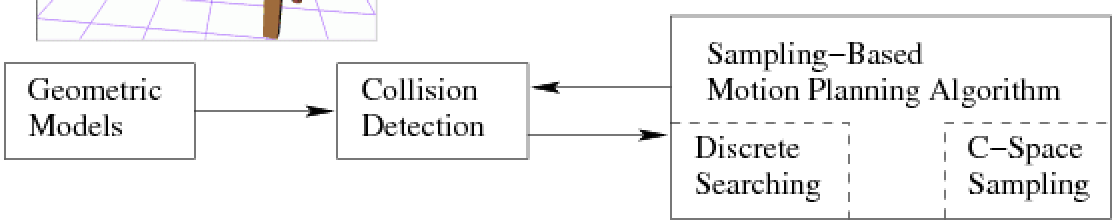
Probabilistic Roadmaps

- Motion planning in complex configuration spaces
- Avoid difficulties in explicit geometric representations
- One solution: Sampling based methods

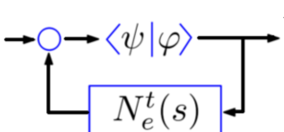








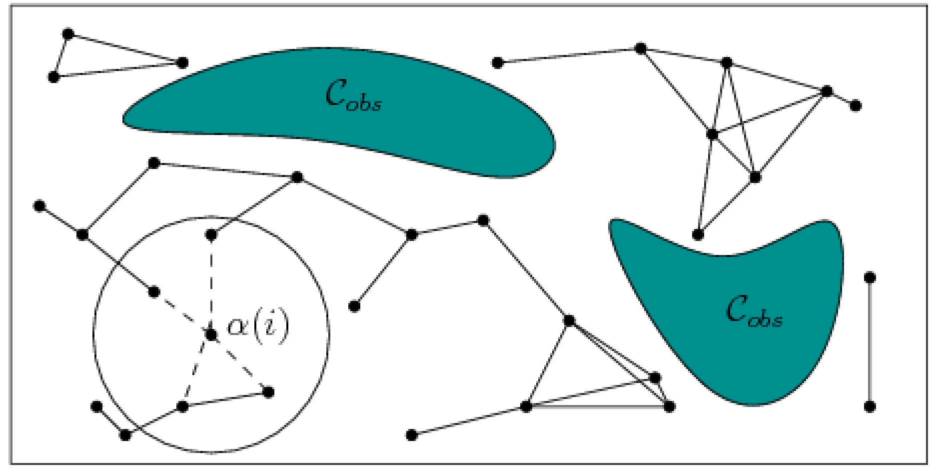
Ref. *Planning Algorithms* by Steve LaValle, 2006



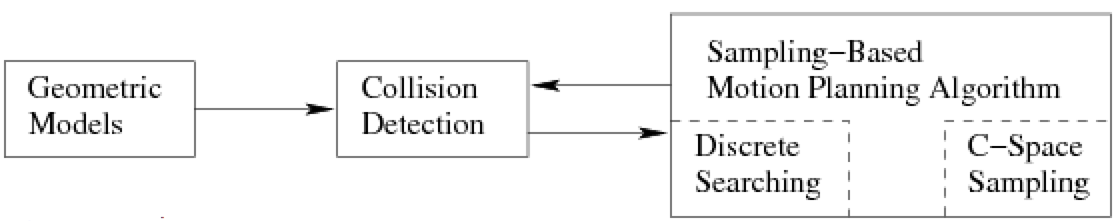
Probabilistic Roadmaps

Steps

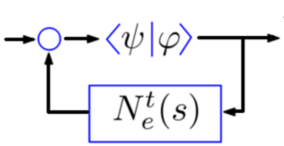
1. Preprocessing phase
2. Query phase




```
BUILD_ROADMAP
G.init(); i ← 0;
while i < N
    if α(i) ∈ C_free then
        G.add_vertex(α(i)); i ← i + 1;
        for each q ∈ NEIGHBORHOOD(α(i),G)
            if ((not G.same_component(α(i), q)) and CONNECT(α(i), q)) then
                G.add_edge(α(i), q);
```



Ref. *Planning Algorithms* by Steve LaValle, 2006



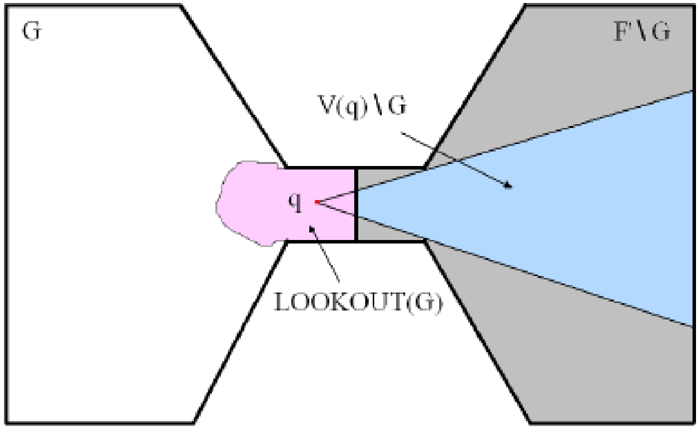
Probabilistic Roadmaps

- How much sampling is needed for a good representation?
- Expansiveness parameterized by $(\alpha, \beta, \epsilon)$

$$LOOKOUT(G) = \{q \in G \mid \mu(\mathcal{V}(q)\backslash G) \geq \beta\mu(\mathcal{F}'\backslash G)\}$$

$$\mu(V(p)) \geq \epsilon \times \mu(F)$$

$$\mu(LOOKOUT(G)) \geq \alpha\mu(G)$$

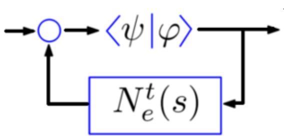


- Failure Probability

$$Pr < \frac{c_1}{\varepsilon\alpha} \exp\left(c_2\varepsilon\alpha(-N + \frac{c_3}{\beta})\right)$$

Ref. Kavraki et al., *Analysis of Probabilistic Roadmaps for Path Planning*, IEEE TRA, 1998.

- Question: Can you estimate $(\alpha, \beta, \epsilon)$?



Spectral Geometry of Spaces

- Kac (1966) posed the problem …
  "Can you hear the shape of a drum?"
- Laplacian operator on manifolds

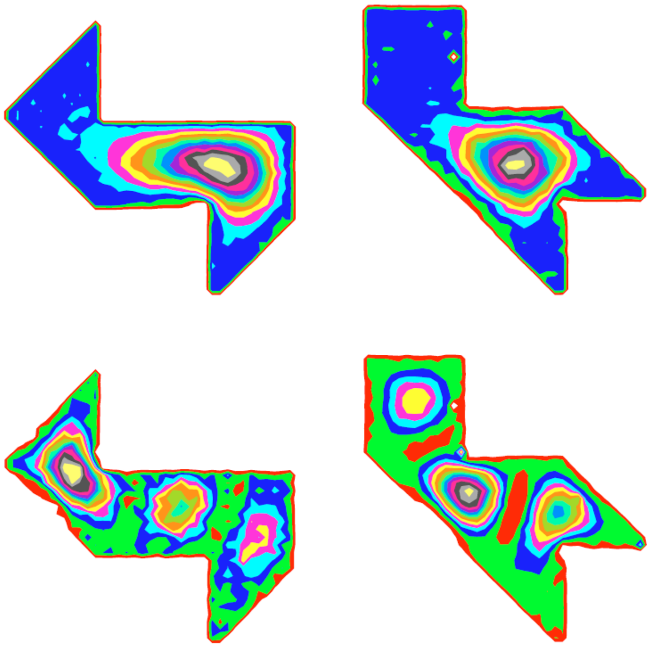
$$\Delta f = \frac{1}{\sqrt{g}} \sum_{j,k} \partial_j \left(g^{jk} \sqrt{g} \partial_k f \right)$$

e.g. $\Delta f = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f$

- Nuemann / Dirichlet problem

$$\Delta \phi = -\lambda \phi$$

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$



[Driscoll, SIAM Rev. 1997]

Spectral Geometry of Spaces

What is known about the spectra?

- Weyl's formula

$$N(\lambda) \sim \mathrm{vol}(M)\mathrm{vol}(B_1^n)\frac{\lambda^{n/2}}{(2\pi^n)}$$

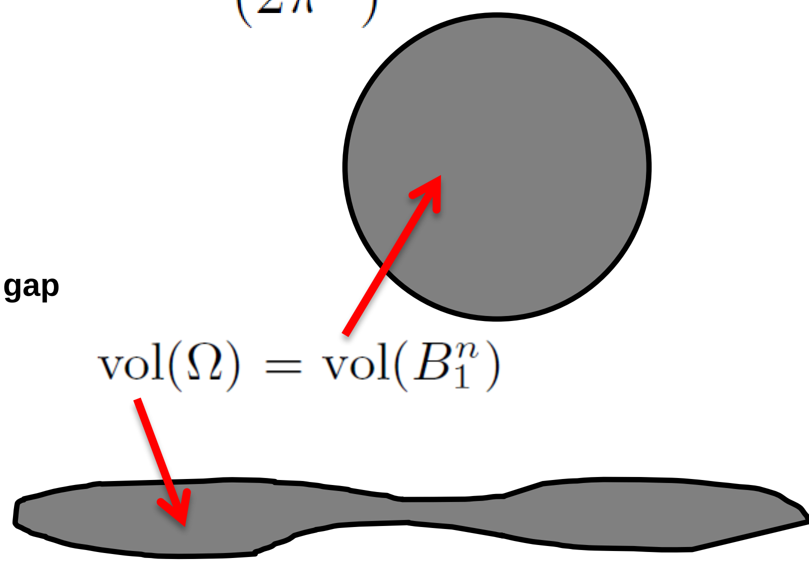
- Rayleigh conjecture

$$\lambda_1(\Omega) \leq \lambda_1(B_1^n)$$

Spectral gap

- Payne-Polya-Weinberger

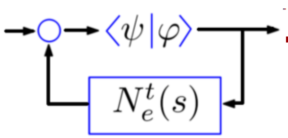
$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(B_1^n)}{\lambda_1(B_1^n)}$$



- Curvature bound

$$\lambda_1(\Omega) \geq -(n-1)^2\kappa/4$$

- Cheeger's inequality $\lambda_1(M) \geq \frac{h^2(M)}{4}.$

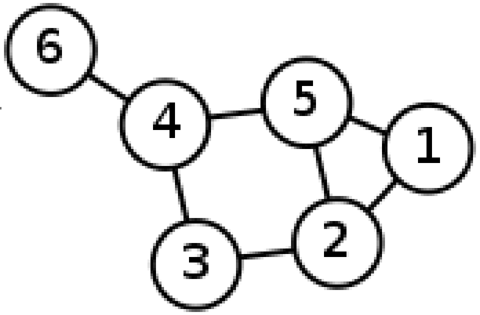


Spectra of PRM graphs

- Adjacency matrix   A
- Degree Matrix   D
- Laplacian Matrix   L = D-A

$$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

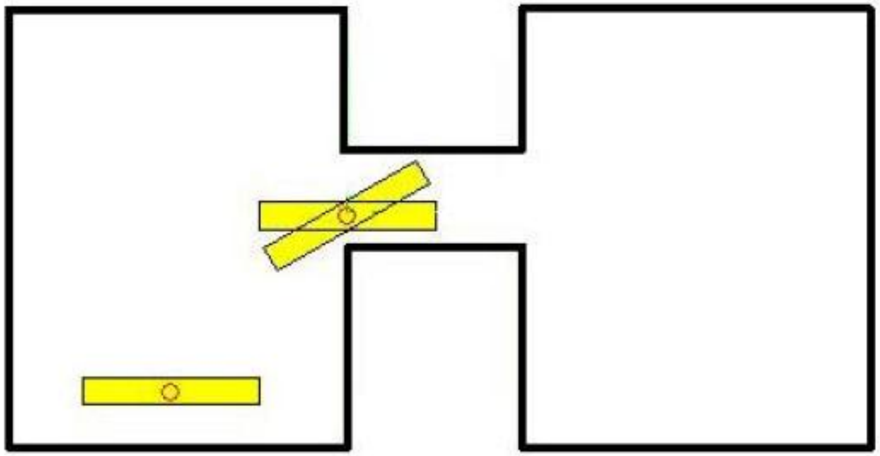
$$\ell_{i,j} := \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise.} \end{cases}$$

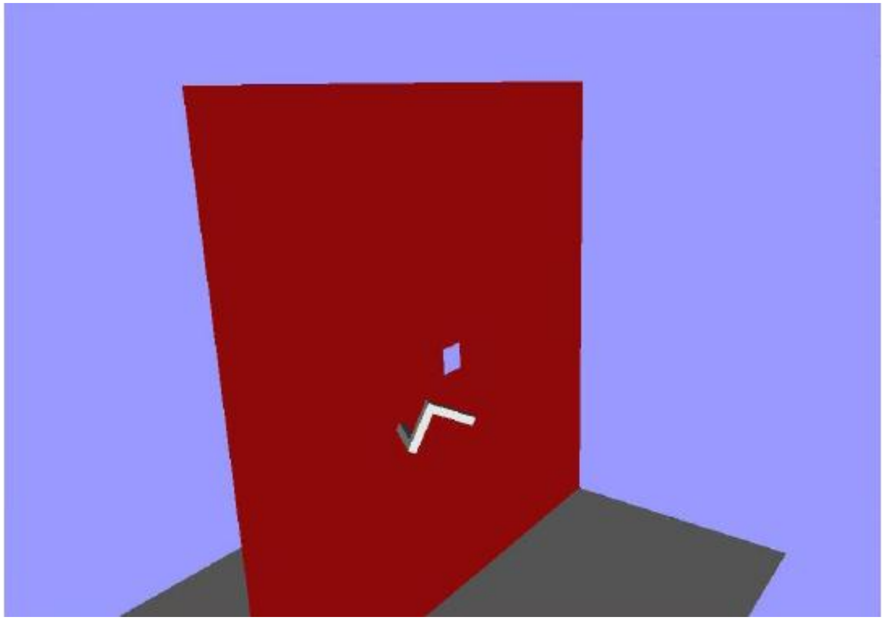


- A discrete version of Laplacian operator
- Spectrum calculation is simple linear algebra

$$L\psi_i = \lambda_i \psi_i \qquad \lambda_1(L) = \inf_{\omega \in 1^{\perp}} \left(\frac{\omega^T L \omega}{\omega^T \omega} \right)$$

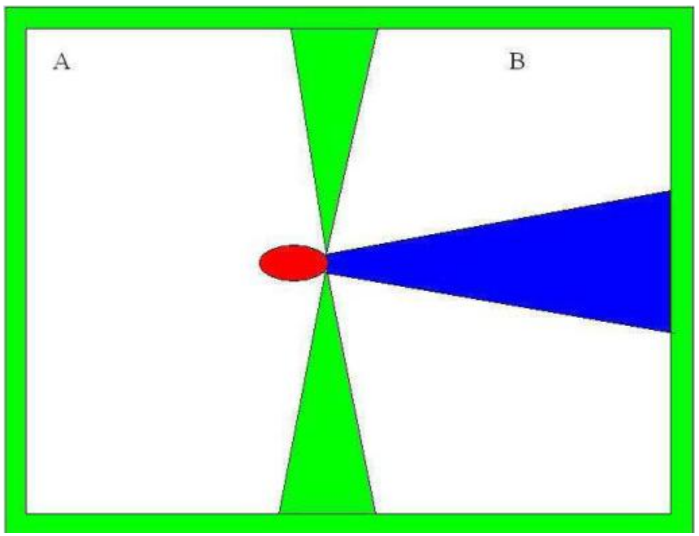
Controlled Experiments



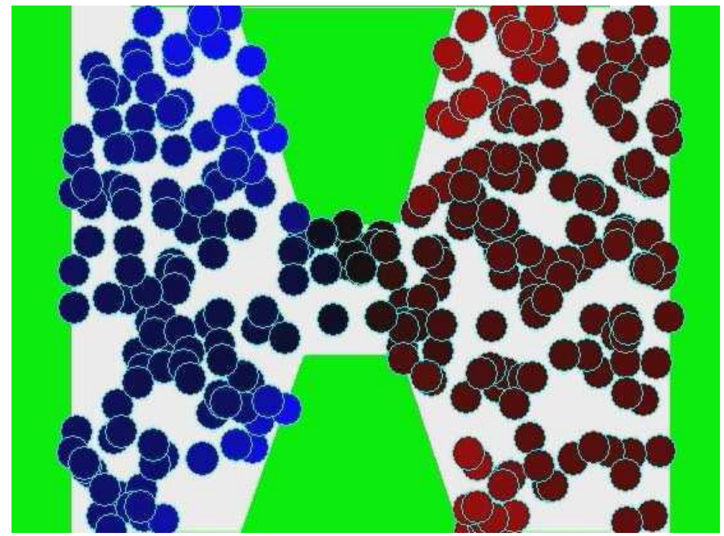


Simulations

Obs 1: Size of Narrow Passage ~ 2nd Lowest Eigenvalue

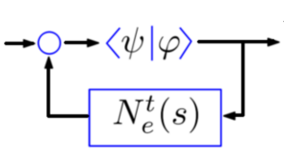


Expansiveness (standard def)



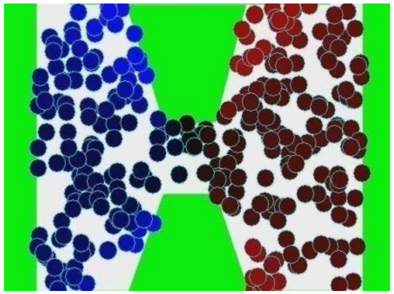
Spectral expansiveness (2nd Lowest Eigenvalue = 0.21).

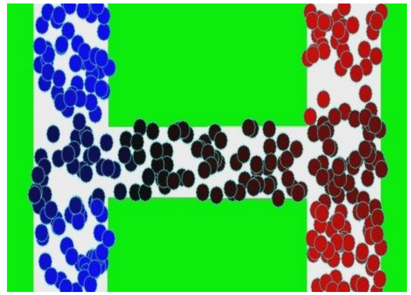
Plot of 2nd eigenvector: Red (+ve), Blue (-ve), Color change at neck.

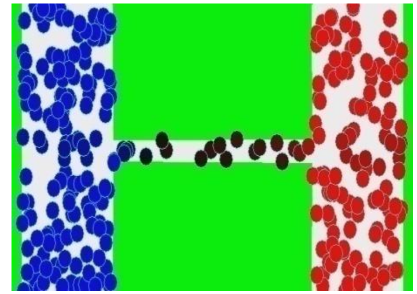


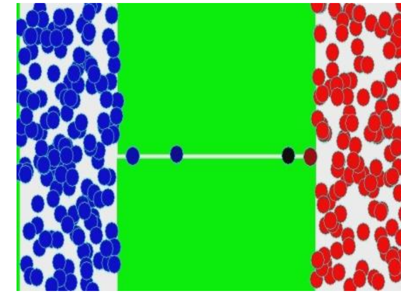
Simulations

Spectral Expansiveness


Small Obstruction


Bigger Obstruction

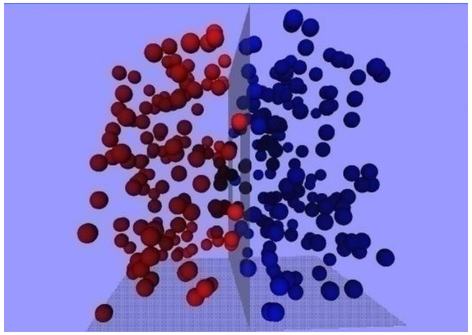

Biggest Obstruction


Almost two parts

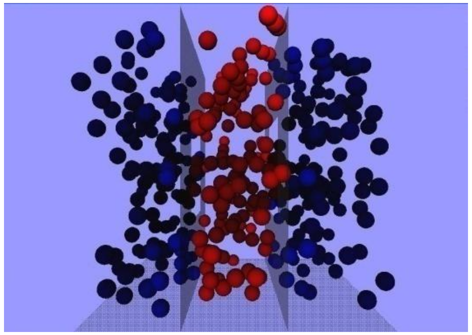
Size of Obstruction	2nd Eigenvalue
No Obstruction	1
Small	0.21
Bigger	0.0865
Biggest	0.013158
Almost two parts	0.000358
Fully Obstructed	0

Simulations

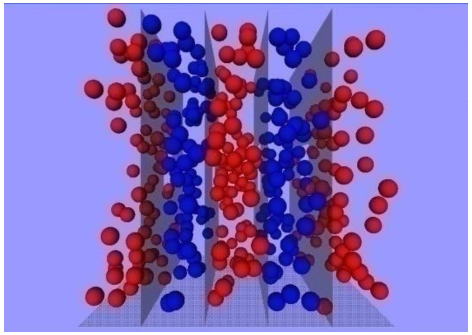
Obs 2: Number of Narrow Passages ~ Spectral Jump



One Obstruction

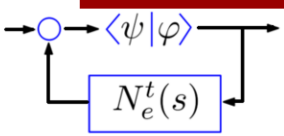


Two Obstruction



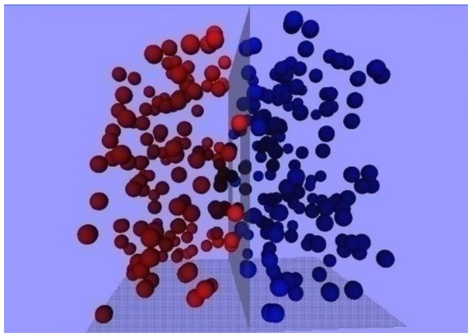
Four Obstruction

Eigenvalues	Single Obstruction	2 Obstruction	4 Obstructions
1st	0	0	0
2nd	21.8647	5.4205	1.6533
3rd	169.036	15.1546	4.5350
4th	171.785	113.7700	7.0097
5th	172.0000	114.4230	9.2204
6th	174.701	119.0040	37.0506

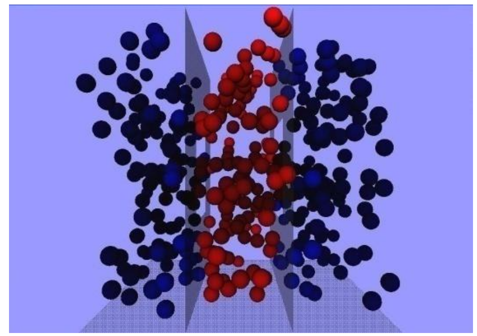


Simulations

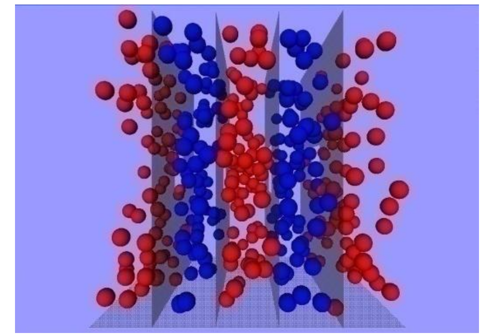
Obs 2: Number of Narrow Passages ~ Spectral Jump



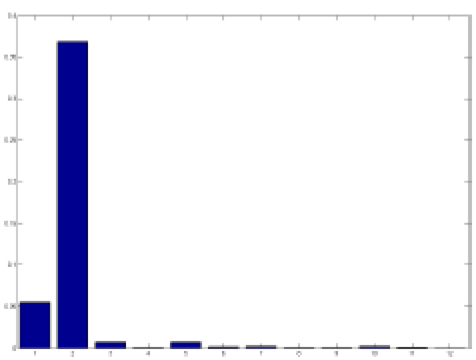
One Obstructions

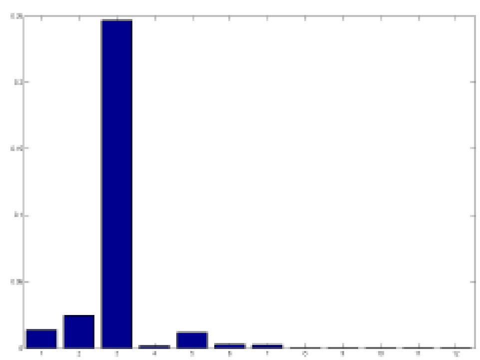


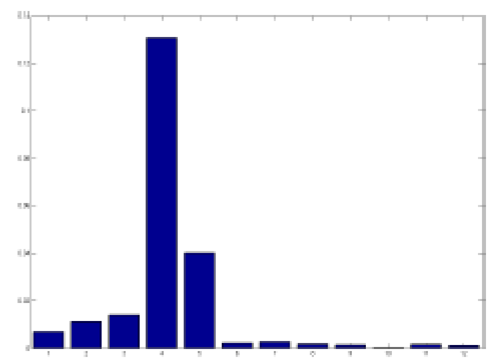
Two Obstructions

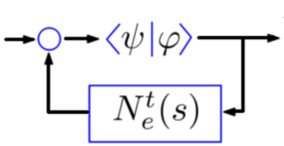


Three Obstructions







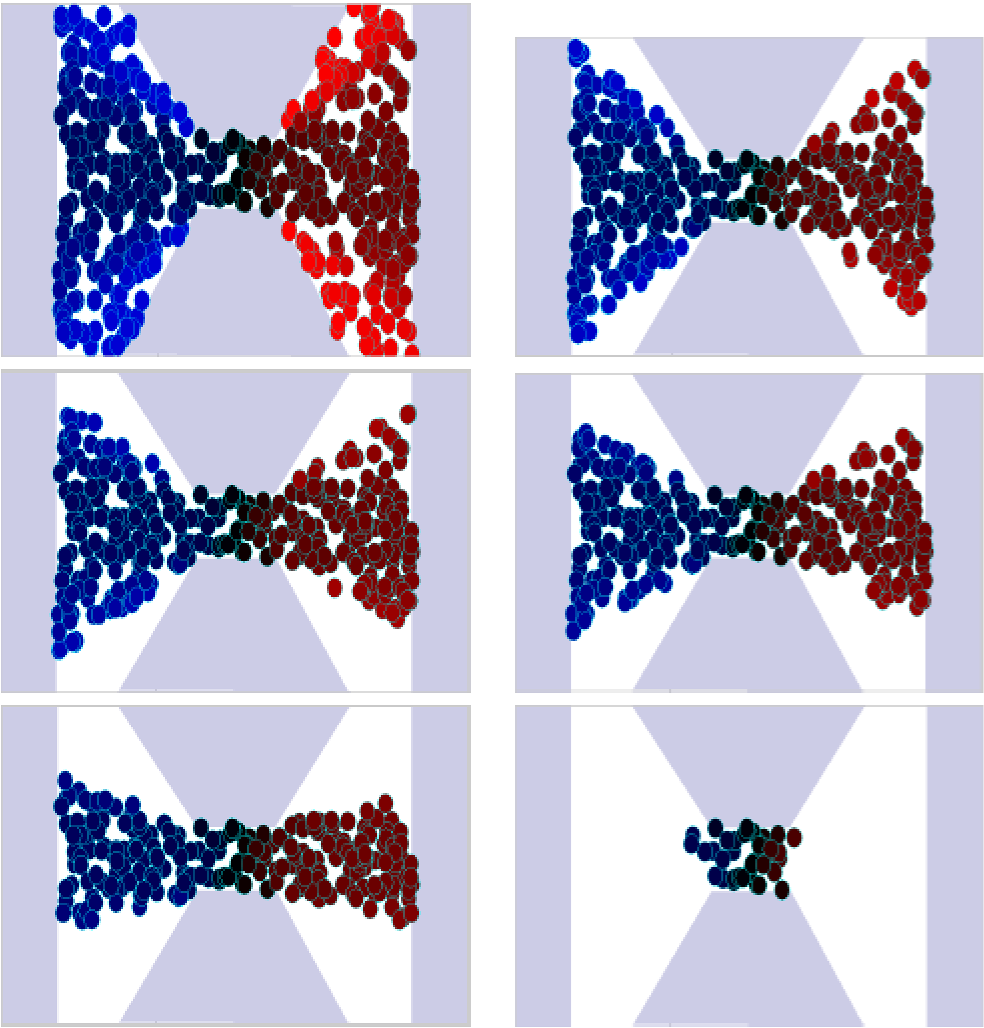


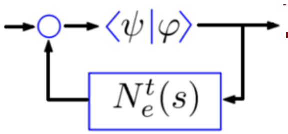
Configuration Labeling

- Node Visibility
  ~ 1/eigenvector scale

- Detect narrow passages
  by appropriate threshold

- Visualization method for
  high dimensional CSpace





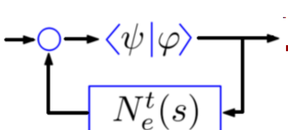
A concrete relationship?

	Spatial Domain		Frequency Domain
Cspace	$(\alpha, \beta, \epsilon)$	⟷	(Δ, Λ, τ)
	↕		↕
Roadmap Graph	(A, B, E)	⟷	(L, λ, t)

- Mhequb Hayat and Abubakr Muhammad, “Spectral Metrics for Expansive Configuration Spaces” (preprint).

Conclusions

- Defined new spectral metrics to compare motion planning difficulties in configuration spaces
  - Computable with reasonable ease (Linear Algebra)
  - Alternative characterization (Frequency Domain)
- Detection of Narrow passages
- Labeling of “difficult” and “easy” parts of the configuration space



Acknowledgements

- **Ideas**
  - Sue Whitesides (Victoria/McGill)
  - Hassan Mohy-ud-Din (Johns Hopkins/LUMS)

- **Funding**
  - Faculty startup research fund, LUMS, Pakistan
  - Travel grant, Higher Education Commission (HEC), Pakistan
  - IEEE-ICRA travel grant for developing countries