Control Using Higher Order Laplacians in Network Topologies

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ABSTRACT

This paper establishes the proper notation and precise interpretation for Laplacian flows on simplicial complexes. In particular, we have shown how to interpret these flows as time-varying discrete differential forms that converge to harmonic forms. The stability properties of the corresponding dynamical system are shown to be related to the topological structure of the underlying simplicial complex. Finally, we discuss the relevance of these results in the context of networked control and sensing.

I. INTRODUCTION AND MOTIVATION

In this paper, we introduce the concept of higher order combinatorial Laplacians of simplicial complexes within the context of networked sensing and control. During the last few years, the graph Laplacian has been used as a powerful tool that lets the network topology be incorporated directly into the dynamics of a networked dynamical system, e.g. [14], [16], [17], [19], [20]. Using tools from algebraic topology, this paper presents a generalization of these ideas through a generalization of the graph Laplacian. The higher order Laplacians do not exist on graphs, but on higher order combinatorial structures called as simplicial complexes. In particular we will study the family of dynamical systems

$$\frac{\partial \omega}{\partial t} = -\Delta_k \omega, \qquad k \geq 0,$$

where Δ_k is the k-th order Laplacian operator, and w in an element of the exterior algebra of certain vector spaces on a simplicial complex.

The motivation for studying this dynamical system comes from some recent work [4], [10], [12] in which researchers have characterized various properties of a networked system such as coverage and routing by certain topological properties of the network. These research efforts have a strong resonance with programs in computational algebraic topology, discrete differential geometry and discrete geometric mechanics [5], [6], [11]. The null spaces of the higher order Laplacians have some very deep connections with the computation of topological invariants. As these null-spaces correspond to the equilibrium points of the above-mentioned family of dynamical systems, the computation of topological invariants is related to the convergence properties of these dynamical systems. The approach taken in this paper is desirable for related research questions in networked sensing and control as it indicates towards devising more quantitative methods and implementation using distributed computation.

Since we use certain concepts from algebraic topology that are not familiar in the systems and networks community, we elaborate on the precise mathematical interpretation of this dynamical system. Specifically, we summarize the properties of simplicial complexes, their homology & cohomology groups and their relation to harmonic analysis via combinatorial Hodge theory. Using this language we interpret the higher order Laplacians as operators on *discrete differential forms*. We also make the connection between the theory of differential forms for differential manifolds and the calculus for discrete forms. We will see that this connection provides a powerful interpretation that lets transport the more familiar ideas in the continuous setting of differential manifolds to the discrete setting of simplicial complexes. We also insist that the theory of combinatorial Laplacians exist independently of the continuous setting and should not always be thought as an *approximation*.

This paper is organized as follows. We outline some basic concepts of simplicial complexes and their topological invariants in Section II. In Section III we introduce the reader to basic Hodge theory. Next, we work out a detailed

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example to elaborate on these concepts. We conclude this section by giving an interpretation of these results in light of the more familiar theory of differential forms on Riemannian manifolds. This mathematical machinery is used in Section IV to study Laplacian flows on simplicial complexes. We conclude by summarizing our observations in Section V.

II. Simplicial Complexes and their Topological Invariants

We first need to introduce some basic tools from algebraic topology for generalizing the concepts of algebraic graph theory that will subsequently be used for designing control networked algorithms. (For a thorough treatment of this subject, see for example [2], [13], [18].)

A. Simplicial Complexes

Graphs can be generalized to more expressive combinatorial objects known as a simplicial complexes. Simplicial complexes are a class of topological spaces that are made of simplices of various dimensions. Given a set of points V, a k-simplex is an unordered subset $\{v_0, v_1, \ldots, v_k\}$ where $v_i \in V$ and $v_i \neq v_j$ for all $i \neq j$. The *faces* of this k-simplex consist of all $(k-1)$-simplices of the form $\{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k\}$ for $0 \leq i \leq k$. A *simplicial complex* is a collection of simplices which is closed with respect to the inclusion of faces. Graphs are a concrete example, where the vertices of the graph correspond to V and edges correspond to 1-simplices. The orderings of the vertices correspond to an *orientation*. A k-simplex $\{v_0, \ldots, v_k\}$ together with an order type is an *oriented* k-simplex denoted by $[v_0, \ldots, v_k]$ (see Figure 1 for typical examples of simplices in dimension zero through three), where a change in orientation corresponds to a change in the sign of the coefficient:

$$[v_0, \ldots, v_j, \ldots, v_i, \ldots, v_k] = -[v_0, \ldots, v_i, \ldots, v_j, \ldots, v_k].$$

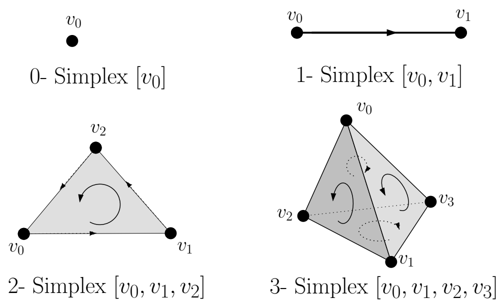

Fig. 1. Oriented simplices of dimension zero through three.

We also generalize the concepts of adjacency and degree for graphs. Two k-simplices σ_i and σ_j of a simplicial complex X are *upper adjacent* if both are faces of some $k+1$-simplex in X. We denote this adjacency by $\sigma_i \frown \sigma_j$. The *upper degree* of a k-simplex, denoted $\deg_u(\sigma)$, is the number of $k+1$-simplices in X of which σ is a face. Now, give X an orientation and suppose $\sigma_i \frown \sigma_j$ with a common $k+1$-simplex ξ. If the orientations of σ_i and σ_j agree with the ones induced by ξ then σ_i, σ_j are said to be *similarly oriented* with respect to ξ. If not, we say that the simplices are *dissimilarly oriented*. In a similar fashion, we also define *lower adjacency* and *lower degree* of simplices. Two k-simplices σ_i and σ_j of a simplicial complex X are *lower adjacent* if both have a common face. We denote this by $\sigma_i \smile \sigma_j$. The *lower degree* of a k-simplex, denoted $\deg_l(\sigma)$, is equal to the number of faces in σ.

B. Exterior Algebra of a Vector Space and Boundary Operators

Let V be a vector space spanned by $\{x_1, \cdots, x_N\}$. Denote by $x_i \wedge x_j = x_i \otimes x_j - x_j \otimes x_i$. Let the space of all such (exterior) products be denoted by $\bigwedge^1(V)$. Similarly, for any $k > 1$, we can find the space $\bigwedge^k(V) \simeq V \otimes V \cdots \otimes V / \sim$ by the appropriate anti-symmetrization operation. It is endowed with an exterior product $\wedge$, which satisfies for all $u, v \in V$,

1) $v \wedge v = 0$.
2) $v \wedge u = -u \wedge v$.

3) $v_1 \wedge v_2 \cdots \wedge v_k = 0$ whenever $v_1, \cdots, v_k$ are linearly independent.
This is the *exterior algebra* of a vector space V. A *boundary operator* $\partial_k : \bigwedge^k(V) \to \bigwedge^{k-1}(V)$ can be defined as

$$\partial_k(v_1 \wedge v_2 \cdots \wedge v_k) = \sum_{j=0}^{n}(-1)^j(v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots v_k),$$

where $\widehat{v_j}$ denotes omission from the product. We see below, how exterior algebras and boundary operators can be defined on a simplicial complex.

C. Chain Complexes and Homology

For each $k \geq 0$, denote by $C_k(X)$, the vector space whose basis is the set of oriented k-simplices of X. For k larger than the dimension of X, $C_k(X) = 0$. The elements of these vector spaces are called as *chains*, which are linear combinations of the basis elements. The *boundary map* is defined to be the linear transformations $\partial_k : C_k(X) \to C_{k-1}(X)$ which acts on basis elements $[v_0, \ldots, v_k]$ via

$$\partial_k[v_0, \ldots, v_k] := \sum_{i=0}^{k}(-1)^i[v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k],$$

as illustrated in Figure 2. These boundary maps gives rise to a so-called *chain complex*, which can be defined as a

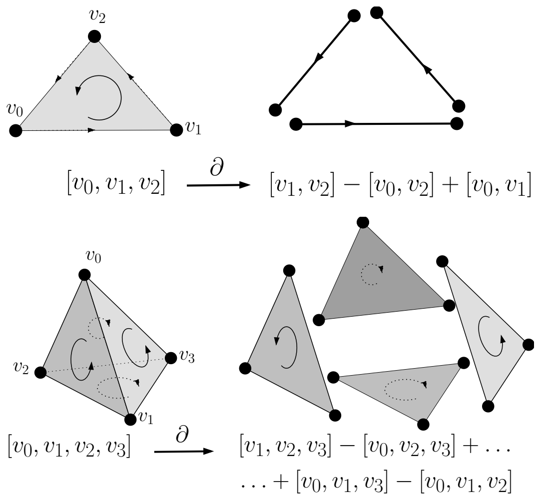

Fig. 2. The boundary operator on a 2-simplex [top] and a 3-simplex [bottom].

sequence of vector spaces and linear transformations

$$0 \to C_n(X) \to \cdots \xrightarrow{\partial_{k+1}} C_k(X) \xrightarrow{\partial_k} C_{k-1}(X) \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \to 0.$$

When dealing with a finite simplicial complex X, the vector spaces $C_i(X)$ are also of finite dimension. Therefore we can easily represent the boundary maps $\partial_i : C_i(X) \to C_{i-1}(X)$ in matrix form. By doing a simple calculation of $\partial_k\partial_{k-1}$ using Equation 1, it is easy to see that the composition $C_k(X) \xrightarrow{\partial_k} C_{k-1}(X) \xrightarrow{\partial_{k-1}} C_{k-2}(X)$ is zero. From this, it follows that im $\partial_k \subset \ker \partial_{k-1}$. The *k-th homology group* of the space X is defined as

$$H_k(X) = \ker \partial_k / \text{im } \partial_{k+1}.$$

Homology groups are used to distinguish topological spaces from one another by identifying the number of ‘holes’ of various dimension, contained in these spaces. Each non-trivial homology class in a certain dimension helps identify a corresponding hole in that dimension. Crudely speaking, the dimension of $H_0(X)$ is the number of connected components (0-dimensional holes) of X. the dimension of $H_1(X)$ is the number of non-contractable cycles in X. For a surface, it identifies the 'punctures' in that surface. For a graph it is the number of loops or circuits. $H_2(X)$ identifies the number of 3-dimensional voids in a space and so on.

D. *Co-chains and Cohomology*

If the vector spaces $C_i(X)$ are defined over $\mathbb{R}$, then one can give an inner product structure to each $C_i(X;\mathbb{R})$ that makes the basis orthogonal.[1] This allows us to define the dual space of the vector space $C_i(X;\mathbb{R})$. A member of this dual space is a real-valued map from the chains to $\mathbb{R}$. These maps are the *co-chains* and the dual space $\mathrm{Hom}_{\mathbb{R}}(C_i(X;\mathbb{R}),\mathbb{R})$ is denoted by $C^i(X;\mathbb{R})$ for dimension i. Since chains are linear combinations of simplices, each map is completely described by specifying its value on each simplex. By duality, we can identify $C^i(X;\mathbb{R})$ with $C_i(X;\mathbb{R})$ via the inner product. It is therefore possible to define a dual operator, which is called as the *co-boundary map*

$$\delta_i : C^i(X;\mathbb{R}) \to C^{i+1}(X;\mathbb{R}).$$

δ_i is is simply the adjoint ∂^*_{i+1} of the boundary map ∂_{i+1}. If $\phi \in C^i(X)$, then for a simplex $\sigma = [v_0, \cdots, v_{i+1}] \in C_{i+1}(X)$

$$\delta_i\phi(\sigma) = \sum_{j=0}^{i+1}(-1)^j \phi([v_0, \cdots, v_{j-1}, v_{j+1}, \cdots .v_{i+1}])$$

In other words, the map $\delta_i\phi$ when evaluated on the simplex σ, is equal to the sum of the evaluations of ϕ on all faces of σ. Since $\delta_i = \partial^*_{i+1}$, the matrix representation of the co-boundary operator for finite simplicial complexes is simply a matrix transpose of the corresponding boundary operator.

Note that the co-boundary maps satisfy $\delta_i\delta_{i+1} = 0$. Therefore, the co-boundary operators also form a chain complex from which the so-called cohomology groups can be obtained. More precisely, the chain complex

$$0 \leftarrow C^n(X;\mathbb{R}) \leftarrow \cdots \xleftarrow{\delta_k} C^k(X;\mathbb{R}) \xleftarrow{\delta_{k-1}} C^{k-1}(X;\mathbb{R}) \cdots \xleftarrow{\delta_1} C^1(X;\mathbb{R}) \xleftarrow{\delta_0} C^0(X;\mathbb{R}) \leftarrow 0$$

gives rise to the cohomology groups

$$H^k(X;\mathbb{R}) = \ker \delta_k / \mathrm{im}\ \delta_{k-1}.$$

When the coefficients in homology and cohomology are chosen over $\mathbb{R}$ (or any other field), the cohomology groups are the exact dual to the homology groups, as a consequence of the universal coefficient theorem [13]. In other words,

$$H^k(X;\mathbb{R}) \simeq \mathrm{Hom}_{\mathbb{R}}(H_k(X;\mathbb{R}),\mathbb{R}). \tag{1}$$

This means that given a cohomology class in the dual space, it is possible to find the corresponding unique homology class.

III. Higher Order Laplacians and Hodge Theory

A. *Combinatorial k-Laplacians*

Using the above definitions of the boundary and co-boundary operators, one can define the operators $\Delta_k : C^k(X;\mathbb{R}) \to C^k(X;\mathbb{R})$ and $\mathcal{L}_k : C_k(X;\mathbb{R}) \to C_k(X;\mathbb{R})$ by

$$\Delta_k = \delta_{k-1}\delta^*_{k-1} + \delta^*_k\delta_k,$$

$$\mathcal{L}_k = \partial_{k+1}\partial^*_{k+1} + \partial^*_k\partial_k.$$

where the operators have simple matrix representations and $\delta^*_k = \partial_{k+1}$. It is easy to see that both operators are representations of the same transformation, one in the original space $C_k(X;\mathbb{R})$ and the other in its dual space $C^k(X;\mathbb{R})$. Therefore, from the point of view of a matrix computation, both representations can be used interchangeably without

[1] Other fields such as $\mathbb{Q}, \mathbb{Z}_p, \mathbb{C}$ can also work.

any ambiguity. We stress on this point since we will use $\mathcal{L}_k$ on chains $C_k(X;\mathbb{R})$ for explicit computations, but will interpret the results on co-chains in $C^k(X;\mathbb{R})$ for interpretation and visualization.

These operators were introduced by Eckmann in 1945 [9], and has been studied since then under the name of *combinatorial Laplacians* [8], [11], [21]. Eckmann, in fact proved an analog of Hodge theory for harmonic differnential forms on Riemannian manifolds, by proving a similar construction for simplicial complexes. He noted that each $C_k(X;\mathbb{R})$ decomposes into orthogonal subspaces

$$C_k(X;\mathbb{R}) = \mathcal{H}_k(X) \oplus \operatorname{im}(\partial_{k+1}) \oplus \operatorname{im}(\partial_k^*),$$

where

$$\mathcal{H}_k(X) = \{c \in C_k(X) : \mathcal{L}_k c = 0\} = \ker \mathcal{L}_k.$$

From this follows that for each k, there is an isomorphism

$$H_k(X;\mathbb{R}) \cong \mathcal{H}_k(X).$$

This means that in order to compute the homology (or the dual cohomology) groups of a simplicial complex, it is enough to study the null space of the matrix $\mathcal{L}_i$. The eigenvectors of $\mathcal{L}_i$ corresponding to the zero eigenvalues are the representative cycles (or cocycles) of a particular homology (or cohomology) class. The set $\{\mathcal{H}_k(X)\}$ has a deep connection with harmonic analysis. Any $c \in C_i(X;\mathbb{R})$ is called *harmonic* if $\mathcal{L}_i c = 0$. Similarly its dual $\omega \in C^i(X;\mathbb{R})$ is called harmonic if $\Delta_i \omega = 0$. The harmonic cocycles are natural analogs of harmonic differential forms on Riemannian manifolds. We discuss this connection shortly.

As mentioned earlier, the boundary operators and their adjoints have matrix representations. In other words, we can also give a matrix representation to the Laplacian. Through this matrix representation, it can be seen that the more familiar *graph Laplacian* is synonymous with $\mathcal{L}_0 : C_0(X) \to \mathcal{C}_0(X)$. Since there are no simplices of negative dimension, $C_{-1}(X)$ is assumed to be 0. Also, the maps ∂_0 and ∂_0^* are assumed to be zero maps so that

$$\mathcal{L}_0 = \partial_1 \partial_1^*.$$

The boundary map $\partial_1 : C_1(X) \to C_0(X)$ maps edges to vertices and its matrix representation is exactly equal to the more familiar incidence matrix B of a graph. Therefore we have $\mathcal{L}_0 = BB^T$, which is the graph Laplacian from algebraic graph theory. It is easy to see that

$$(\mathcal{L}_0)_{ij} = \begin{cases} \deg(v_i), & \text{if } i = j, \\ -1, & \text{if} v_i \sim v_j, \\ 0, & \text{otherwise,} \end{cases}$$

where $\deg(v_i)$ (degree of a vertex in a graph) is the same as $\deg_u(v_i)$ and the adjacency relation $v_i \sim v_j$ (adjacency of two vertices in an edge) is the same as $v_i \frown v_j$ defined previously. One can also write $\mathcal{L}_0$ in the familiar averaging formula,

$$\mathcal{L}_0(v_i) = \sum_{v_i \frown v_j} (v_i - v_k).$$

Let us generalize these representations for $k > 0$. Let $\sigma_1, \cdots, \sigma_n$ be the oriented k-simplices of X. We observe that

$$(\partial_{k+1}\partial_{k+1}^*)_{ij} = \begin{cases} \deg_u(\sigma_i) & \text{if } i = j, \\ 1 & \text{if } i \neq j,\ \sigma_i \frown \sigma_j \text{ with dissimilar orientation,} \\ -1 & \text{if } i \neq j,\ \sigma_i \frown \sigma_j, \text{ with similar orientation,} \\ 0 & \text{otherwise.} \end{cases}$$

Similarly,

$$(\partial_k^* \partial_k)_{ij} = \begin{cases} k+1 & \text{if } i = j, \\ 1 & \text{if } i \neq j,\ \sigma_i \smile \sigma_j \text{ with similar orientation,} \\ -1 & \text{if } i \neq j,\ \sigma_i \smile \sigma_j, \text{ with dissimilar orientation,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $\sigma_i \frown \sigma_j$, then it is always the case that $\sigma_i \smile \sigma_j$ with an opposite orientation similarity. Therefore, whenever $\sigma_i \frown \sigma_j$, $(\partial_k^* \partial_k)_{ij} + (\partial_{k+1}\partial_{k+1}^*)_{ij} = 0$. Combining these observations, we get

$$(\mathcal{L}_k)_{ij} = (\partial_k^* \partial_k)_{ij} + (\partial_{k+1}\partial_{k+1}^*)_{ij} = \begin{cases} \deg_u(\sigma_i) + k + 1 & \text{if } i = j, \\ 1 & \text{if } i \neq j,\ \sigma_i \not\frown \sigma_j, \text{ and} \\ & \sigma_i \smile \sigma_j \text{ with similar} \\ & \text{orientation,} \\ -1 & \text{if } i \neq j,\ \sigma_i \not\frown \sigma_j, \text{ and} \\ & \sigma_i \smile \sigma_j \text{ with dissimilar} \\ & \text{orientation,} \\ 0 & \text{if } i \neq j \text{ and either } \sigma_i \frown \sigma_j \\ & \text{or } \sigma_i \not\smile \sigma_j. \end{cases}$$

Let $\epsilon_{ij} \in \{-1, 1\}$ capture the similarity or dissimilarity of orientation between simplices σ_i, σ_j, then the formula for the k-Laplacian can be explicitly written at the simplex level as,

$$\mathcal{L}_k(\sigma_i) = (\deg_u(\sigma_i) + k + 1)\sigma_i + \sum_{\sigma_i \smile \sigma_j} \epsilon_{ij}\sigma_j - \sum_{\sigma_i \frown \sigma_m} \epsilon_{im}\sigma_m. \tag{2}$$

Finally, if n_k is the number of k-simplices in X, $D = \text{diag}(\deg_u(\sigma_1), \cdots, \deg_u(\sigma_{n_k}))$, and A_u, A_l denote the upper and lower adjacency matrices between the k-simplices respectively, then

$$\mathcal{L}_k = D - A_u + (k+1)I_{n_k} + A_l, \quad k > 0.$$

For $k = 0$, we have the familiar matrix decomposition from algebraic graph theory,

$$\mathcal{L}_0 = D - A_u.$$

B. Examples

We now give a simple example to illustrate these formulae. Consider the simplicial complex K given in Figure 3. The boundary operators are given below.

$\partial_1 =$

[12]	[13]	[23]	[24]	[34]	[35]	[45]	
−1	−1	0	0	0	0	0	[1]
1	0	−1	−1	0	0	0	[2]
0	1	1	0	−1	−1	0	[3]
0	0	0	1	1	0	−1	[4]
0	0	0	0	0	1	1	[5]

$\partial_2 =$

[2, 3, 4]	
0	[1, 2]
0	[1, 3]
1	[2, 3]
−1	[2, 4]
1	[3, 4]
0	[3, 5]
0	[4, 5]

The Laplacians are computed as

$$\mathcal{L}_0 = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}, \quad \mathcal{L}_1 = \begin{pmatrix} 2 & 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & -1 & -1 & 0 \\ -1 & 1 & 3 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 3 & 1 & -1 \\ 0 & -1 & -1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -1 & -1 & 1 & 2 \end{pmatrix}, \quad \mathcal{L}_2 = 3.$$

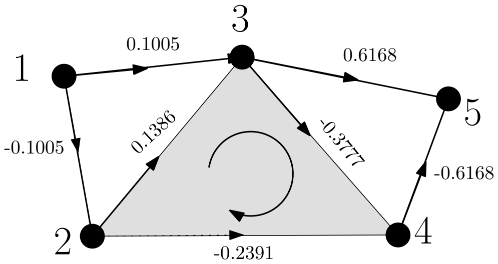

Fig. 3. A simplicial complex K of dimension 2. The numbers labelled on the edges correspond to the corresponding components of one of the two zero-eigenvectors of the 1-Laplacian of K.

Let us compute the eigenvector decomposition of $\mathcal{L}_1$. It turns out that it has two eigenvalues equal to zero. Therefore the null-space has dimension two and is spanned by the vectors

$$v_1 = \begin{pmatrix} -0.1005 \\ 0.1005 \\ 0.1386 \\ -0.2391 \\ -0.3777 \\ 0.6168 \\ -0.6168 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -0.6090 \\ 0.6090 \\ -0.4138 \\ -0.1951 \\ 0.2187 \\ -0.0236 \\ 0.0236 \end{pmatrix}.$$

We interpret these results below. The Hodge decomposition described above tells us that the null-space of $\mathcal{L}_1$ is isomorphic to the first homology group $H_1(K;\mathbb{R})$. Since the dimension of the null-space is two, there are two non-trivial 1-cycles in K. The two eigenvectors corresponding to the zero eigenvalues represent the two homology classes of $H_1(K;\mathbb{R})$. We can interpret the eigenvectors as follows. Let $c \in C_1(K;\mathbb{R})$ be a chain whose real valued coefficients are given by the respective coefficients of v_1.

$$c = -0.1005[12] + 0.1005[13] + 0.1386[23] - 0.2391[24] - 0.3777[34] + 0.6168[35] - 0.6168[45].$$

It is clear that the boundary operator ∂_1 operates on c via $\partial_1 v_1 = 0$. Therefore $\partial_1 c = 0$ and c is a 1-cycle. In fact, it is a non-trivial cycle because $\partial_2^* v_1 = 0$, so $c \notin \text{im } \partial_2$. Hence, c or v_1 represents a homology class of the simplicial complex. In the same way, v_2 is a representation of a second non-trivial homology class of $H_1(K;\mathbb{R})$. To see why v_1 and v_2 represent two different homology classes, observe that v_1 is orthogonal to v_2 by virtue of being eigenvectors. This is consistent with the observation that there are indeed two 'holes' in K, one of which is bounded by the edges [12],[23] and [31], and the other by the edges [34],[45] and [53]. Another important interpretation of the vectors v_1 and v_2 is to consider the co-vectors v_1^*, v_2^* as functions on the 1-simplices (i.e. the edges). As an example, the coefficients of v_1 have been put on their corresponding edges in Figure 3. Identify v_1^*, v_2^* with the maps $\phi_1, \phi_2 : C_1(K;\mathbb{R}) \to \mathbb{R}$, i.e. $\phi_1, \phi_2 \in C^1(K;\mathbb{R})$. Now observe that $\delta_1 \phi_1 = \partial_2^* v_1 = 0$. Similarly $\delta_1 \phi_2 = 0$. This means that ϕ_1, ϕ_2 are both non-trivial cohomology classes of $H^1(K;\mathbb{R})$.

Now take any chain $e \in C_1(K;\mathbb{R})$ such that $\partial_1 e = 0$. If $\phi_1 e = 0$ then e is a trivial cycle, otherwise it bounds a hole in K. As an example, let

$$e = [12] - [13] + [23] \in C_1(K;\mathbb{R}),$$

then $\phi_1 e$ is computed by $v_1^*[1,-1,1,0,0,0,0]^* = -0.1005 - 0.1005 + 0.1386 \neq 0$. In simple words, take any cyclic path on K and accumulate the corresponding values of v_1 along this path, reversing the sign of the coefficients when going against the orientation. If the accumulated sum is equal to zero, the path does not bound a hole in K. If $z = [23] + [34] - [24]$ then $\phi_1 z = -0.3518 - 0.3518 - 0.7082 = 0$.

A more elaborate compuation has been depicted in Figure 4. The simplicial complex has been drawn in the upper left corner, and is made up of 117 nodes, 650 edges and 1393 two-simplices. The null-space of the 1-Laplacian has dimension

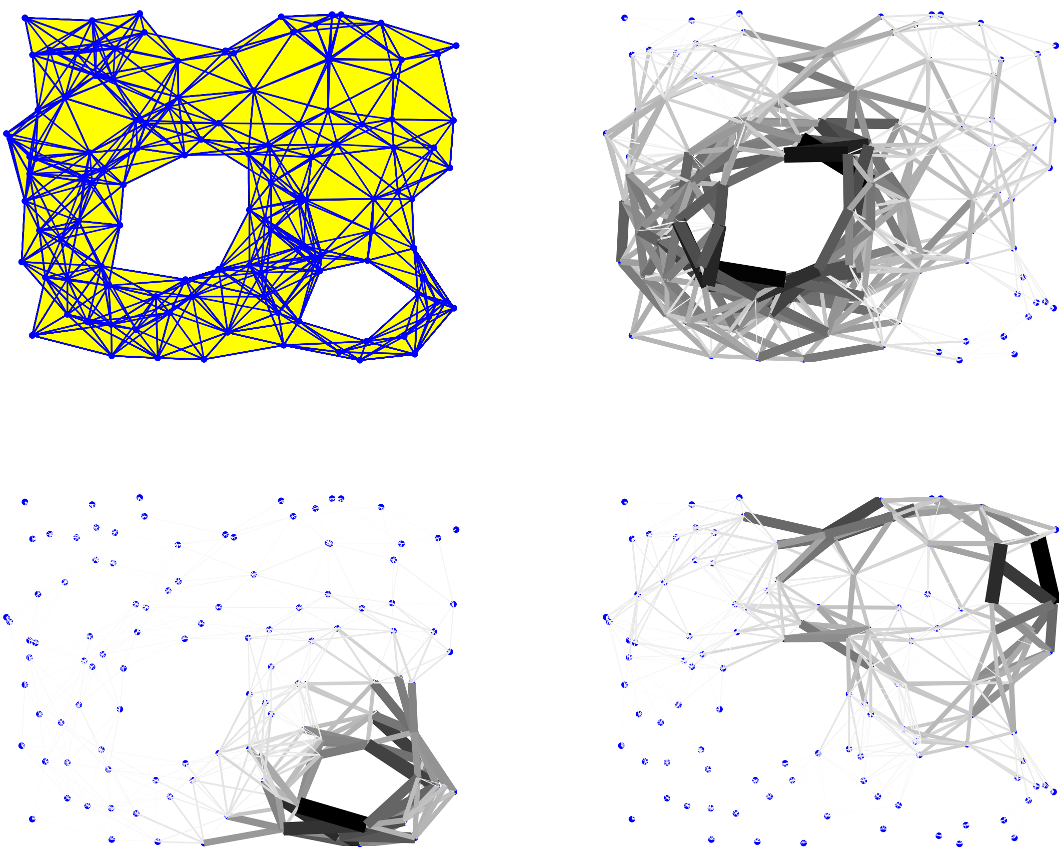

Fig. 4. A simplicial complex [upper left], its Harmonic 1-forms [upper right and lower left] and the 1-form corresponding to the smallest non-zero eigenvalue of $\mathcal{L}_1$ [lower right].

2. The two cohomology classes (or for that matter the homology classes) correspond to the two zero eigenvectors of the Laplacian matrix. The two eigenvectors have been depicted in the upper right and lower left parts of the figure. The thickness of an edge is directly proportional to the magnitude of its corresponding eigenvector component. It can be seen that the magnitudes peak close to the holes and in this case, distinguish the two holes quite clearly. The co-chain drawn in the the lower right of the figure corresponds to the smallest non-zero eigenvalue. An interpretation of these 'near-harmonic' co-chains will be given later.

C. Differential Forms and the Continuous Laplace-Beltrami Operators

Assume that $\omega \in C^k(X;\mathbb{R})$, which is not necessarily a co-cycle. $\delta_1\omega \in C^{k+1}(X;\mathbb{R})$ is a map from the k-simplices in X to $\mathbb{R}$. Let $s = \sum_j \alpha_j \sigma_j \in C^{k+1}(X;\mathbb{R})$ where $\alpha_j \in \mathbb{R}$ and σ_j are the k-simplices. Suppose we wish to do the computation

$$\delta_k\omega(s) = \sum_j \alpha_j.\delta_1\omega(\sigma_j).$$

By duality $\delta_k = \partial^*_{k+1}$, which means that

$$\delta_k \omega = \omega \partial_{k+1}$$

Therefore,

$$\sum_j \alpha_j.\delta_k \omega(\sigma_j) = \sum_j \alpha_j.\omega(\partial_{k+1}\sigma_j) = \omega\partial_{k+1}(\sum_j \alpha_j \sigma_j) = \omega \partial_{k+1} s.$$

This computation can be seen as a discretization of Stoke's theorem for integration over a domain $\mathcal{D}$,

$$\int_{\partial \mathcal{D}} \omega = \int_{\mathcal{D}} d\omega,$$

where $\partial\mathcal{D}$ is the boundary of the domain and $d\omega$ is the *differential* of ω. This suggests that $C^i(X;\mathbb{R})$ can also be interpreted as the set of *discrete differential forms* on a combinatorial manifold X. Let us recall below, how the differential forms are defined on differential manifolds and what is the significance of their cohomology groups.

For Riemannian Manifolds, let $T^*_x M$ be the vector space, called as the cotangent space at $x \in M^n$. It is locally spanned by the differentials $\{dx_1, \cdots, dx_n\}$. Now define the exterior algebra on this vector space by $\bigwedge^k(T^*_x M)$. At each k, $\bigwedge^k(T^*_x M)$ is spanned by all exterior products

$$\underbrace{dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}}_{k-\text{times}}.$$

By varying over $x \in M$, we define the k-differential forms $\Omega^k(M) = \Gamma(\bigwedge^k(T^*M))$. $\Omega^0(M)$ is simply the set of functions from M to $\mathbb{R}$. Each k-form $\omega \in \Omega^k(M)$ can be given in local coordinates by

$$\omega = \sum_{i_1 < \cdots < 1_k} \omega_{i_1 \cdots i_k} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}.$$

There is a linear operator $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$ that locally satisfies

$$d_k \omega = \sum_{i_1 < \cdots < 1_k} \frac{\partial \omega_{i_1 \cdots i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}.$$

It can be verified that $d_{k+1} d_k = 0$. Therefore, we have a chain complex

$$\Omega^n(M) \leftarrow \cdots \xleftarrow{d_k} \Omega^k(M) \xleftarrow{d_{k-1}} \Omega^{k-1}(M) \leftarrow \cdots \xleftarrow{d_1} \Omega^1(M) \xleftarrow{d_0} \Omega^0(M).$$

The cohomology groups associated with this chain complex are the so-called *De-Rham Cohomology* groups defined by

$$H^k_{DR}(M) = \frac{\ker d_k}{\mathrm{im} d_{k-1}}.$$

There is a corresponding co-differential operator $d^*_k = (-1)^{n(k+1)+1} \star d_k \star$, defined via the *Hodge-star operator* $\star : \Omega^k(M) \to \Omega^{n-k}(M)$. The Hodge-star operator satisfies

$$\star(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}},$$

where $\{dx_{i_1}, \cdots, dx_{i_k}, dx_{j_1}, \cdots, dx_{j_{n-k}}\}$ form a (positive) basis of $T^*_x M$.

The *Laplacian-Beltrami operator* on a manifold is defined by

$$\Delta_k = d^*_k d_k + d_{k-1} d^*_{k-1}.$$

It can be shown [15] that for $M = \mathbb{R}^n$, this operator can be explicitly written as

$$\Delta_k \omega = -\sum_{m=1}^{n} \frac{\partial^2 \omega_{i_1 \cdots i_k}}{\partial x_m^2} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

In particular, for $k = 0$, we have the familiar

$$\Delta_0 f = -\sum_{m=1}^{n} \frac{\partial^2 f}{\partial x_m^2} = -\nabla \cdot (\nabla f).$$

on a function $f : M \to \mathbb{R}$. A k-differential form $\omega \in \Omega^k(M)$ is called *harmonic* if

$$\Delta_k \omega = 0.$$

As a direct analog of the Hodge decomposition for simplicial complexes described above, every cohomology class in $H^k_{DR}(M)$ contains precisely one harmonic k-form. It can also be proved that the harmonic k-forms are the ones that minimize the L^2-norm on $\Omega^k(M)$ [15]. Hence, the null-spaces of the Laplacian-Beltrami operators are isomorphic to their respective De-Rham cohomology groups.

Let us now study, how the discretization of these k-forms leads to analog results on simplicial complexes. Consider a simplicial complex X obtained by discretizing a manifold. Triangulations are concrete examples of such a construction for 2-manifolds. $C^0(X; \mathbb{R})$ is the collection of functions on the vertices, which can be seen as the discretization of functions in $\Omega^0(M)$. Let f be one such function. The 1-form $df \in \Omega^1(M)$ can be approximated on two neighbouring points, $x, x + \Delta x \in M$ by

$$df(x) \approx \frac{1}{\Delta x} f(x + \Delta x) - f(x).$$

If v_0, v_1 are the vertex representation of points $x, x + \Delta x$ on X and $[v_0, v_1]$ is a 1-simplex, then

$$\delta_0 f([v_0, v_1]) = f(v_1) - f(v0).$$

This clearly suggests that $\delta_0 f$ is a discretization of df on M. A similar interpretation holds for other k-forms. The calculus of discrete differential forms has been explored in detail by various researchers [5], [6], [7], [11]. In particular, the discretization of the Hodge-star operator has led to consistent definitions of a discrete adjoint operator d^*. We omit these details here as it involves a lengthy digression into duals of simplicial complexes [5], [11]. Using these results, it is possible to interpret the discrete combinatorial Laplacians as discretizations of the respective continuous Laplace-Beltrami operators. In particular,

$$\mathcal{L}_0 = \partial_1 \partial_1^*,$$

is just a discretization of

$$\Delta_0 = -\nabla \cdot (\nabla)$$

A similar discretization holds for the the higher order Laplacians as well. This interpretation has several advantages. Most importantly, it allows us to conceptualize the behavior of the combinatorial operator as an approximation of its continuous counterpart. We conclude with the following observations.

1) It is possible to *geometrically approximate* the higher order Laplace-Beltrami operators on manifolds to their discrete analogs using the calculus of discrete differential forms. Since any simplicial complex is realizable in some $\mathbb{R}^N$ for N large enough [18], it may be considered as an approximation to some continuous sub-manifold in $\mathbb{R}^N$. The approximation however, may be too crude to give a good geometrical insight on the behavior of the combinatorial Laplacians on the simplicial complex.
2) Due to this interpretation, it is more insightful to consider the combinatorial Laplacians as operators on cochains (interpreted as discrete k-forms), rather than operators on chains of simplices.
3) The combinatorial k-Laplacians, however exist on simplicial complexes, independent of such a conceptual discretization as shown by Eckmann [9]. Therefore, it is not necessary to see the behavior of the combinatorial Laplacians in light of a discretization only.

IV. Laplacian Flows on Simplicial Complexes

The graph Laplacian is a powerful tool that allows the network topology to be directly incorporated into the equations of a networked dynamical system. The most important application of this approach has been seen in consensus problems, where a simple averaging law such as

$$\dot{x_i}(t) = -\sum_{v_i \sim v_j} (x_i(t) - x_j(t)),$$

can be re-written as

$$\frac{dc(x,j)}{dt} = -\mathcal{L}_0 c(x,j),$$

where the component operator is given through $c(x,j) = [x_{1,j}, \cdots, x_{N,j}]^T$. For a connected graph, it can be shown through the spectral properties of $\mathcal{L}_0$ that all states converge towards a common state. We study below a generalization of this dynamical system for higher order Laplacians. Let the time flow of a discrete time-varying k-form $\omega(t)$ be given by $\Delta_k \omega : \mathbb{R}^+ \times C^k(X;\mathbb{R}) \to C^k(X;\mathbb{R})$. We study the dynamical system

$$\frac{\partial \omega(t)}{\partial t} = -\Delta_k \omega(t), \qquad \omega(0) = \omega_0 \in C^k(X;\mathbb{R}). \tag{3}$$

The equilibrium points of this dynamical system is the set of k-forms given by

$$\{\omega \in C^k(X;\mathbb{R}) \mid \Delta_k \omega = 0\} = \ker(\Delta_k).$$

Let us recall a few definitions of stability of a dynamical system. A dynamical system $\dot{x}(t) = f(x(t))$ is called *Lyapunov stable*, if for every initial condition $x(0)$, $x(t)$ is bounded for all $t \geq 0$. It is *semi-stable*, if for every initial condition $x(0)$, $\lim_{t\to\infty} x(t)$ exists. If is called *asymptotically stable*, if for every initial condition $x(0)$, $\lim_{t\to\infty} x(t) = 0$. Asymptotic stability is stronger than semi-stability, and semi-stability is stronger than Lyapunov stability.

An eigenvalue λ of a linear operator A is called *semi-simple* if its algebraic multiplicity (the number of times the eigenvalue repeats in the spectrum) is equal to its geometric multiplicity given by $\dim(\ker(\lambda I - A))$. It can be shown that a *linear dynamical system* [1] is Lyapunov stable if all of its eigenvalues have either a negative real part, or they are purely imaginary and semi-simple. It is semi-stable, if its eigenvalues have a negative real part or they are semi-simple zeros. It is asymptotically stable if all eigenvalues have a negative real part. We observe the following result.

Proposition IV-A: The dynamical system of Equation 3 is semi-stable.

Proof: Let there be n_k number of k-simplices in X. Let the eigenvalues of Δ_k be given by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n_k}$. As $\delta_{k-1}\delta^*_{k-1} \succeq 0$ and $\delta^*_k \delta_k \succeq 0$, it follows that $\Delta_k = \delta_{k-1}\delta^*_{k-1} + \delta^*_k\delta_k \succeq 0$. Therefore, all of its eigenvalues are real and $\lambda_i \geq 0$ for all $i \leq n_k$. If the null-space $\mathcal{M}$ has dimension $\beta_k = 0$, then all eigenvalues of the dynamical system are real-negative, and therefore the system is asymptotically stable, which also implies semi-stability.

If $\beta > 0$, then the algebraic multiplicity of $\lambda_1 = 0$ is the same as its geometric multiplicity $\dim(\ker(\lambda_1 I - \Delta_k)) = \beta_k$. Therefore, all eigenvalues of the system in Equation 3 are either negative or semi-simple zero. This implies that the system is always, at least semi-stable. ■

Note that the condition that $\beta_k = 0$ is equivalent to saying that the k-th cohomology group (or the respective k-th homology group) is zero. Therefore, we have the following corollary.

Corollary IV-B: The system in Equation 3 is asymptotically stable if and only if $H^k(X;\mathbb{R}) = 0$.

This proves that for any initial $\omega(0) \in C^k(X;\mathbb{R})$, the trajectory $\omega(t), t \geq 0$ always converges to some point in $\ker \Delta_k$. Therefore the dynamical system is a mechanism for producing discrete harmonic k-forms on simplicial complexes from any arbitary k-forms. Snapshots of a simulation of one such flow has been given in Figure 6.

It should be noted however that the dynamical system produces the harmonic k-form only in the limit. We would therefore like to *estimate* how close the trajectory at time t is to the limit. Let $\omega_\infty = \lim_{t\to\infty} \omega(t)$ for the system initially at $\omega(0)$. The measurable quantity $\|\partial\omega(t)/\partial t\| = \|\Delta_k \omega(t)\|$ helps us with this estimation, where $\|.\|$ is the usual L^2-norm as we are only dealing with simplicial complexes that have a finite number of simplices in each dimension. Note that $\Delta_k \omega_\infty = 0$. We have

$$\|\Delta_k \omega(t)\| = \|\Delta_k \omega(t) - \Delta_k \omega_\infty\| \leq \|\Delta_k\| \|\omega(t) - \omega_\infty\|.$$

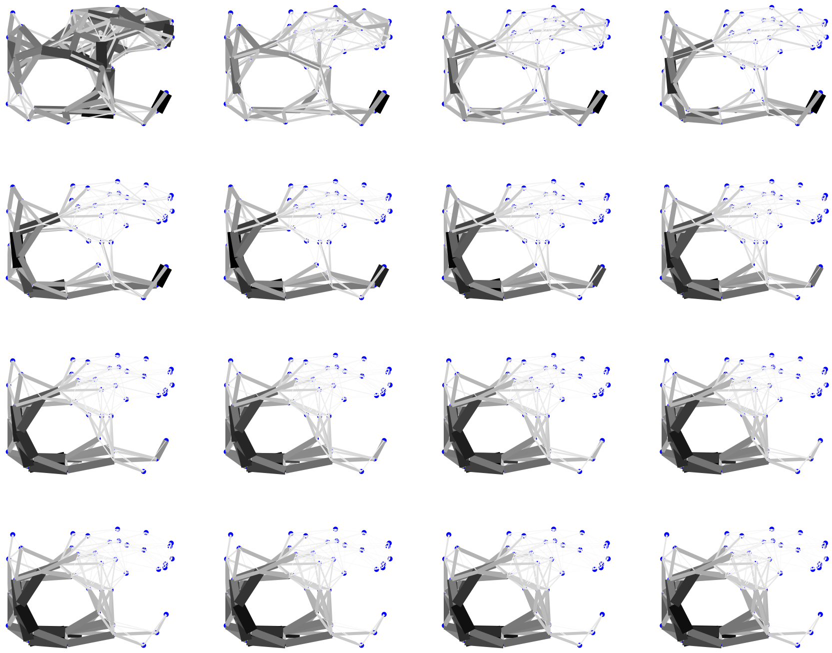

Fig. 5. The 1-Laplacian flow on a simplicial complex with one non-trivial (co)homology class. The flow stabilizes to a harmonic 1-form that accumulates high energy on the edges close to the hole.

Therefore,

$$\|\omega(t) - \omega_\infty\| \geq \frac{\|\Delta_k \omega(t)\|}{\|\Delta_k\|}.$$

Since $\|\Delta_k\|^2 = \sum_{i=1}^{n_k} \lambda_i^2$, for t large enough, the convergence is dominated by the smallest non-zero eigenvalue λ_+. Hence

$$\|\omega(t) - \omega_\infty\| \geq \frac{1}{\lambda_+}\|\Delta_k \omega(t)\|. \tag{4}$$

This gives us an upper bound. To obtain a lower bound, note that

$$\omega(t) = \exp(-\Delta_k t)\omega(0).$$

By the semi-stability of the system, $\lim_{t\to\infty} \exp(-\Delta_k t)$ exists. Furthermore, since the eigenvalues are real,

$$\exp(-\Delta_k t) = \sum_{k=1}^{n_k} \exp(-\lambda_k t) v_k v_k^*,$$

where $v_k \in C^k(X;\mathbb{R})$ are the eigenvectors of the operator Δ_k. In the limit $\exp(-\lambda_k t) \to 0$ if $\lambda_k > 0$, and $\exp(-\lambda_k t) =$

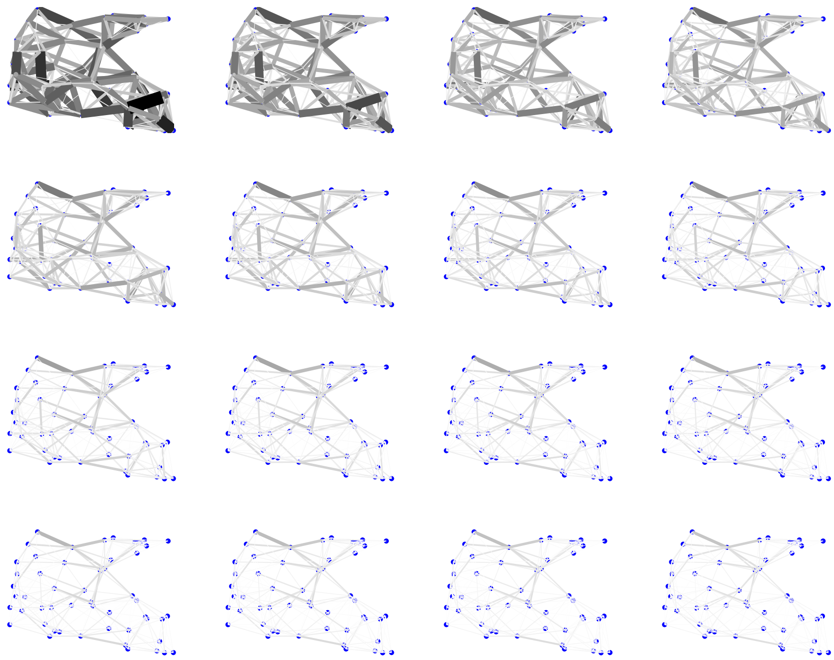

Fig. 6. The 1-Laplacian flow on a simplicial complex with trivial (co)homology in dimension one. The corresponding dynamical system is asymptotically stable, resulting in zero energy in the limit.

1 for all t if $\lambda_k = 0$. Combining these observations, we see that

$$\begin{aligned} \|\omega(t) - \omega_\infty\| &= \left\| \exp(-\Delta_k t)\omega(0) - \lim_{t\to\infty} \exp(-\Delta_k t)\omega(0) \right\|, \\ &\leq \left\| \sum_{k=\beta_k+1}^{n_k} v_k \exp(-\lambda_k t) v_k^* \right\| \|\omega(0)\|, \\ &= \left(\sum_{k=\beta_k+1}^{n_k} \exp(-2\lambda_k t) \right)^{1/2} \|\omega(0)\|. \\ &\leq (n_k - \beta_k - 1)\exp(-\lambda_+ t)\|\omega(0)\|. \end{aligned} \tag{5}$$

Once again, we note that the convergence is dominated by the smallest positive eigenvalue λ_+. Typically, the number of simplices n_k is much bigger than the number of holes β_k in the simplicial complex. Therefore, combining the inequalities 4 and 5, we have

$$\frac{1}{\lambda_+}\|\Delta_k \omega(t)\| \leq \|\omega(t) - \omega_\infty\| \leq n_k \exp(-\lambda_+ t)\|\omega(0)\|. \tag{6}$$

These bounds give an estimate of how close $\omega(t)$ is to being harmonic. This is demonstrated in the graph of Figure 7, which shows the decay of $\|\Delta_k \omega(t)\|$ in the simulation of Figure 6. We say that the k-form $\omega(t)$ is ε-harmonic at time

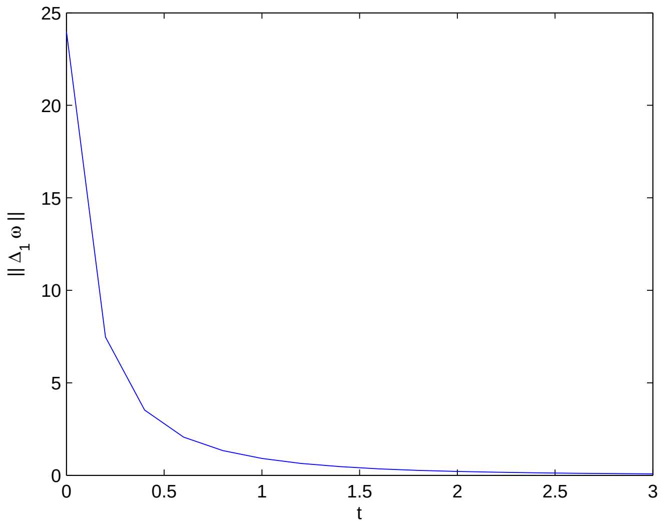

Fig. 7. The graph of $\|\Delta_1\omega(t)\|$ against time shows an asymptotic convergence to zero.

t, if $\|\omega(t) - \omega_\infty\| \leq \varepsilon$. Therefore, $\omega(t)$ is ε-harmonic for all

$$t \geq t_\varepsilon = \frac{1}{\lambda_+} \ln \left(\frac{\varepsilon}{n_k \|\omega(0)\|} \right).$$

For the special case when $H^k(X;\mathbb{R}) = 0$, the system stabilizes at zero, i.e. $\omega_\infty = 0$. We say that the k-form $\omega(t)$ is ε-trivial, if $\|\omega(t)\| \leq \varepsilon$ for all $t \geq t_\varepsilon$.

V. Discussion and Future Work

In the context of relevant problems in networked sensing and control, this paper provokes the following questions and observations.

1) *Decentralized computation:* Equation 2 suggests that the Laplacian flow can be implemented in a distributed manner at the simplex level, involving only those simplices that are upper or lower adjacent to a given simplex. This is desirable from the point of scalability and efficiency in networked systems and may emerge as a decentralized method of computing homology/cohomology groups. Question remains however, as to how to make this method independent of the initial conditions of the dynamical system, so as to detect *all* homology classes of the network.
2) *Near-harmonic forms and quantitative topology:* The operation of the Laplacian on a co-chain indicates how close it is to being a cohomology class. This lets one to quantify the fragilities of a simplicial complex. The eigenvectors corresponding to small non-zero eigenvalues indicate what parts of the complex are close to becoming holes. This is a more useful abstraction of network holes than merely knowing their absence or presence. Work needs to be done in order to quantify these notions more carefully.
3) *Global Vs. Local:* Empirical evidence suggests that the energy distribution of the harmonic forms is related to whether the holes in a simplicial complex are 'big' or 'small'. The smaller the hole, the more concentrated is the energy on the simplices close to the hole. Again, this is useful for distinguishing between local and global features of a network. However, the relationship is more complex than this as it also depends on the density of simplices within various parts of the complex. The above mentioned interpretation is useful only under the assumption of a uniform density.

4) *Consensus at simplex level:* The graph Laplacian has emerged as an important tool for studying consensus algorithms at the node level. The generalization presented in this paper suggests that the Laplacian flows could be useful for establishing consensus at the edge-level or at a higher simplex level.

The further investigation of these questions is a subject of our current research.

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