EE 561: Digital Control Systems

Quiz 2 Solution

Marks: 10 + 3 Bonus — *Wed, Nov* 13, *Fall* 2013.

- **Problem 1**
  The discrete-time state space pair $(\mathbf{\Phi}, \mathbf{\Gamma})$ is said to be controllable if for any initial state $\mathbf{x}_0$ and any final state $\bar{\mathbf{x}}$, there exists a finite sequence of controls inputs $u_0, u_1, \ldots, u_{n-1}$ that transfers $\mathbf{x}_0$ to $\bar{\mathbf{x}}$.

1. Prove that the controllability of $(\mathbf{\Phi}, \mathbf{\Gamma})$ is equivalent to the invertibility of the matrix

$$\mathcal{C} = \begin{bmatrix} \mathbf{\Gamma} & \mathbf{\Phi\Gamma} & \mathbf{\Phi^2\Gamma} & \ldots & \mathbf{\Phi^{n-1}\Gamma} \end{bmatrix}.$$

Hint: The solution of a discrete-time state space difference equation is given by

$$\mathbf{x}_n = \mathbf{\Phi}^n \mathbf{x}_0 + \sum_{m=0}^{n-1} \mathbf{\Phi}^{n-m-1} \mathbf{\Gamma} u_m. \tag{1}$$

According to the definition of controllability, the pair $(\mathbf{\Phi}, \mathbf{\Gamma})$ is said to be controllable if for any given initial state $\mathbf{x}_0$ and final state $\bar{\mathbf{x}} = \mathbf{x}_n$, we can find a finite sequence of controls inputs $u_0, u_1, \ldots, u_{n-1}$ that transfers $\mathbf{x}_0$ to $\mathbf{x}_n$. This means that $\mathbf{x}_0$ and $\mathbf{x}_n$ are known, while the sequence of controls inputs $u_0, u_1, \ldots, u_{n-1}$ is the unknown. Separating the knowns and the unknowns in (1) to either side of the equation, we get

$$\sum_{m=0}^{n-1} \mathbf{\Phi}^{n-m-1} \mathbf{\Gamma} u_m = \mathbf{x}_n - \mathbf{\Phi}^n \mathbf{x}_0.$$

Now we expand the summation on the LHS as

$$\mathbf{\Phi}^{n-1}\mathbf{\Gamma} u_0 + \mathbf{\Phi}^{n-2}\mathbf{\Gamma} u_1 + \ldots + \mathbf{\Phi\Gamma} u_{n-2} + \mathbf{\Gamma} u_{n-1} = \mathbf{x}_n - \mathbf{\Phi}^n \mathbf{x}_0,$$

and rearrange it

$$\mathbf{\Gamma} u_{n-1} + \mathbf{\Phi\Gamma} u_{n-2} + \ldots + \mathbf{\Phi}^{n-2}\mathbf{\Gamma} u_1 + \mathbf{\Phi}^{n-1}\mathbf{\Gamma} u_0 = \mathbf{x}_n - \mathbf{\Phi}^n \mathbf{x}_0.$$

Assuming a single control input, the summation on the LHS can be separated into a product of the $n \times n$ matrix $\begin{bmatrix} \mathbf{\Gamma} & \mathbf{\Phi\Gamma} & \ldots & \mathbf{\Phi}^{n-1}\mathbf{\Gamma} \end{bmatrix}$ and the $n \times 1$ vector $\mathbf{u} = [u_{n-1} \ \ u_{n-2} \ \ \ldots \ \ u_0]^\top$ as

$$\begin{bmatrix} \mathbf{\Gamma} & \mathbf{\Phi\Gamma} & \ldots & \mathbf{\Phi}^{n-2}\mathbf{\Gamma} & \mathbf{\Phi}^{n-1}\mathbf{\Gamma} \end{bmatrix} \begin{bmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix} = \mathbf{x}_n - \mathbf{\Phi}^n \mathbf{x}_0.$$

The matrix $\begin{bmatrix} \mathbf{\Gamma} & \mathbf{\Phi\Gamma} & \ldots & \mathbf{\Phi}^{n-1}\mathbf{\Gamma} \end{bmatrix} = \mathcal{C}$, and let $\mathbf{x}_n - \mathbf{\Phi}^n \mathbf{x}_0 = \boldsymbol{\chi}$, an $n \times 1$ vector. This leads to

$$\mathcal{C}\mathbf{u} = \boldsymbol{\chi},$$
$$\implies \mathbf{u} = \mathcal{C}^{-1}\boldsymbol{\chi}.$$

Now it follows from the definition of controllability that for any $\mathbf{x}_0$ and $\mathbf{x}_n$, an input vector $\mathbf{u}$ containing the sequence $u_0, u_1, \ldots, u_{n-1}$ exists if and only if $\mathcal{C}$ is invertible.

2. Prove that the invertibility of $\mathcal{C}$ is equivalent to the invertibility of the matrix

$$\mathbf{W} = \sum_{m=0}^{n-1} \mathbf{\Phi}^m \mathbf{\Gamma}\mathbf{\Gamma}^\top (\mathbf{\Phi}^\top)^m. \tag{2}$$

$$\begin{aligned}
\mathcal{C}\mathcal{C}^\top &= \begin{bmatrix} \mathbf{\Gamma} & \mathbf{\Phi\Gamma} & \dots & \mathbf{\Phi}^{n-1}\mathbf{\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{\Gamma} & \mathbf{\Phi\Gamma} & \dots & \mathbf{\Phi}^{n-1}\mathbf{\Gamma} \end{bmatrix}^\top \\
&= \begin{bmatrix} \mathbf{\Gamma} & \mathbf{\Phi\Gamma} & \dots & \mathbf{\Phi}^{n-1}\mathbf{\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{\Gamma}^\top \\ (\mathbf{\Phi\Gamma})^\top \\ \vdots \\ (\mathbf{\Phi}^{n-1}\mathbf{\Gamma})^\top \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{\Gamma} & \mathbf{\Phi\Gamma} & \dots & \mathbf{\Phi}^{n-1}\mathbf{\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{\Gamma}^\top \\ \mathbf{\Gamma}^\top\mathbf{\Phi}^\top \\ \vdots \\ \mathbf{\Gamma}^\top \left(\mathbf{\Phi}^{n-1}\right)^\top \end{bmatrix}.
\end{aligned}$$

Now because $\left(\mathbf{\Phi}^{n-1}\right)^\top = (\mathbf{\Phi}^\top)^{n-1}$, it follows that

$$\begin{aligned}
\mathcal{C}\mathcal{C}^\top &= \mathbf{I}\mathbf{\Gamma}\mathbf{\Gamma}^\top\mathbf{I} + \mathbf{\Phi}\mathbf{\Gamma}\mathbf{\Gamma}^\top\mathbf{\Phi}^\top + \ldots + \mathbf{\Phi}^{n-1}\mathbf{\Gamma}\mathbf{\Gamma}^\top(\mathbf{\Phi}^\top)^{n-1} \\
&= \sum_{m=0}^{n-1} \mathbf{\Phi}^m \mathbf{\Gamma}\mathbf{\Gamma}^\top (\mathbf{\Phi}^\top)^m = \mathbf{W}.
\end{aligned}$$

$$\begin{gathered}
\text{If } \det(\mathcal{C}) \neq 0, \\
\text{then } \det(\mathcal{C}\mathcal{C}^\top) = \det(\mathbf{W}) \neq 0, \\
\text{because } \det(\mathcal{C}\mathcal{C}^\top) = \det(\mathcal{C})\det(\mathcal{C}^\top) = \det(\mathcal{C})^2, \\
\text{from the fact that } \det(\mathcal{C}) = \det(\mathcal{C}^\top).
\end{gathered}$$