• Problem 1
The discrete-time state space pair \((\Phi, \Gamma)\) is said to be controllable if for any initial state \(x_0\) and any final state \(\bar{x}\), there exists a finite sequence of controls inputs \(u_0, u_1, \ldots, u_{n-1}\) that transfers \(x_0\) to \(\bar{x}\).

1. Prove that the controllability of \((\Phi, \Gamma)\) is equivalent to the invertibility of the matrix

\[
C = [\Gamma \ \Phi \Gamma \ \Phi^2 \Gamma \ \ldots \ \Phi^{n-1} \Gamma].
\]

Hint: The solution of a discrete-time state space difference equation is given by

\[
x_n = \Phi^n x_0 + \sum_{m=0}^{n-1} \Phi^{n-m-1} \Gamma u_m.
\]

According to the definition of controllability, the pair \((\Phi, \Gamma)\) is said to be controllable if for any given initial state \(x_0\) and final state \(\bar{x} = x_n\), we can find a finite sequence of controls inputs \(u_0, u_1, \ldots, u_{n-1}\) that transfers \(x_0\) to \(x_n\). This means that \(x_0\) and \(x_n\) are known, while the sequence of controls inputs \(u_0, u_1, \ldots, u_{n-1}\) is the unknown. Separating the knowns and the unknowns in (1) to either side of the equation, we get

\[
\sum_{m=0}^{n-1} \Phi^{n-m-1} \Gamma u_m = x_n - \Phi^n x_0.
\]

Now we expand the summation on the LHS as

\[
\Phi^{n-1} \Gamma u_0 + \Phi^{n-2} \Gamma u_1 + \ldots + \Phi \Gamma u_{n-2} + \Gamma u_{n-1} = x_n - \Phi^n x_0,
\]

and rearrange it

\[
\Gamma u_{n-1} + \Phi \Gamma u_{n-2} + \ldots + \Phi^{n-2} \Gamma u_1 + \Phi^{n-1} \Gamma u_0 = x_n - \Phi^n x_0.
\]

Assuming a single control input, the summation on the LHS can be separated into a product of the \(n \times n\) matrix \([\Gamma \ \Phi \Gamma \ \ldots \ \Phi^{n-1} \Gamma]\) and the \(n \times 1\) vector \([u_{n-1} \ u_{n-2} \ \ldots \ u_0]^\top\)

as

\[
\begin{bmatrix}
\Gamma & \Phi \Gamma & \ldots & \Phi^{n-2} \Gamma & \Phi^{n-1} \Gamma
\end{bmatrix}
\begin{bmatrix}
u_{n-1} \\
u_{n-2} \\
\vdots \\
u_1 \\
u_0
\end{bmatrix} = x_n - \Phi^n x_0.
\]

The matrix \([\Gamma \ \Phi \Gamma \ \ldots \ \Phi^{n-1} \Gamma] = C\), and let \(x_n - \Phi^n x_0 = \chi\), an \(n \times 1\) vector. This leads to

\[
Cu = \chi,
\]

\[\Rightarrow u = C^{-1} \chi.
\]

Now it follows from the definition of controllability that for any \(x_0\) and \(x_n\), an input vector \(u\) containing the sequence \(u_0, u_1, \ldots, u_{n-1}\) exists if and only if \(C\) is invertible.
2. Prove that the invertibility of $C$ is equivalent to the invertibility of the matrix

$$W = \sum_{m=0}^{n-1} \Phi^m \Gamma \Gamma^T (\Phi^T)^m. \quad (2)$$

$$CC^T = [\Gamma \Phi \Gamma \ldots \Phi^{n-1}\Gamma] [\Gamma \Phi \Gamma \ldots \Phi^{n-1}\Gamma]^T$$

$$= [\Gamma \Phi \Gamma \ldots \Phi^{n-1}\Gamma] \begin{bmatrix} \Gamma^T \\ (\Phi \Gamma)^T \\ \vdots \\ (\Phi^{n-1}\Gamma)^T \end{bmatrix}$$

$$= [\Gamma \Phi \Gamma \ldots \Phi^{n-1}\Gamma] \begin{bmatrix} \Gamma^T \\ \Gamma^T \Phi^T \\ \vdots \\ \Gamma^T (\Phi^{n-1})^T \end{bmatrix}.$$

Now because $(\Phi^{n-1})^T = (\Phi^T)^{n-1}$, it follows that

$$CC^T = \Gamma \Gamma^T I + \Phi \Gamma \Gamma^T \Phi^T + \ldots + \Phi^{n-1} \Gamma \Gamma^T (\Phi^T)^{n-1}$$

$$= \sum_{m=0}^{n-1} \Phi^m \Gamma \Gamma^T (\Phi^T)^m = W.$$

If $\det(C) \neq 0$,

then $\det(CC^T) = \det(W) \neq 0$,

because $\det(CC^T) = \det(C) \det(C^T) = \det(C)^2$,

from the fact that $\det(C) = \det(C^T)$. 

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