EE 561: Digital Control Systems

Midterm Exam's Solution

Wed 9th Oct. *Fall 2013*

- **Problem 1 [34 Points]**

1. Use the z-transform method to solve and plot

$$y_{n+1} + 3y_n = 4\delta_{n-2}, \quad y_0 = 2. \tag{1}$$

Verify your answer for $n = 0, 1, 2$ by solving the equation by ordinary recursion.

Taking the z-transform on both sides of the difference equation,

$$\begin{aligned}
\mathcal{Z}\{y_{n+1}\} &= \sum_{n=0}^{\infty} y_{n+1}z^{-n} = z\sum_{n=0}^{\infty} y_{n+1}z^{-(n+1)} \\
&= z\left(\sum_{n=-1}^{\infty} y_{n+1}z^{-(n+1)} - y_0\right) = z\left(\sum_{k=0}^{\infty} y_k z^{-k} - y_0\right) \\
&= z(Y(z) - y_0). \\
\mathcal{Z}\{3y_n\} &= 3Y(z). \\
\mathcal{Z}\{4\delta_{n-2}\} &= 4z^{-2}.
\end{aligned}$$

Plugging in the z-transforms into the difference equation,

$$\begin{aligned}
z(Y(z) - y_0) + 3Y(z) &= 4z^{-2}, \\
zY(z) - 2z + 3Y(z) &= 4z^{-2}, \\
\implies Y(z) &= \frac{2z + 4z^{-2}}{z+3} = 2\frac{z}{z+3} + 4z^{-3}\frac{z}{z+3}.
\end{aligned}$$

Now, taking the inverse z-transform of $Y(z)$ leads to

$$y_n = 2(-3)^n \mathbb{1}_n + 4(-3)^{n-3}\mathbb{1}_{n-3}.$$

From this expression,

$$\begin{aligned}
y_0 &= 2(-3)^0\mathbb{1}_0 + 4(-3)^{-3}\mathbb{1}_{-3} = 2, \\
y_1 &= 2(-3)^1\mathbb{1}_1 + 4(-3)^{-2}\mathbb{1}_{-2} = -6, \\
y_2 &= 2(-3)^2\mathbb{1}_2 + 4(-3)^{-1}\mathbb{1}_{-1} = 18.
\end{aligned}$$

From difference equation,

$$\begin{aligned}
y_0 &= 2, \\
y_1 &= 4\delta_{-2} - 3y_0 = -6, \\
y_2 &= 4\delta_{-1} - 3y_1 = 18.
\end{aligned}$$

Hence, verified for $n = 0, 1, 2$.

2. Assume that this difference equation is implemented on a DSP processor with analog I/O clocked at 0.1 s. Plot the frequency magnitude response of the sampled data system.

First of all, we need to find the transfer function $H(z)$, by assuming u_n as input and zero initial conditions,

$$\begin{aligned}
zY(z) + 3Y(z) &= U(z), \\
\implies \frac{Y(z)}{U(z)} &= H(z) = \frac{1}{z+3}.
\end{aligned}$$

For the frequency response, after substituting $z = e^{j\omega}$, we get

$$H(e^{j\omega}) = \frac{1}{e^{j\omega}+3} = \frac{1}{(\cos\omega + 3) + j\sin\omega}.$$

$$|H(e^{j\omega})| = \frac{1}{(\cos\omega+3)^2 + \sin^2\omega} = \frac{1}{6\cos\omega + 10}.$$

From the fact that normalized frequency $\omega = \Omega T$, where Ω is the unnormalized frequency and T is the sampling period, we obtain

$$|H(e^{j\Omega T})| = \frac{1}{6\cos(0.1\Omega)+10}.$$

Because $-\pi \le \omega \le \pi$, the range of values of Ω for $T = 0.1$ will be $-10\pi \le \Omega \le 10\pi$. A plot of $|H(e^{j\Omega T})|$ at different values of Ω in this range is shown in Fig. 1

Ω	-10π	$-15\pi/2$	-5π	$-5\pi/2$	0	$5\pi/2$	5π	$15\pi/2$	10π
$\|H(e^{j\Omega T})\|$	0.5	0.4168	0.3162	0.265	0.25	0.265	0.3162	0.4168	0.5

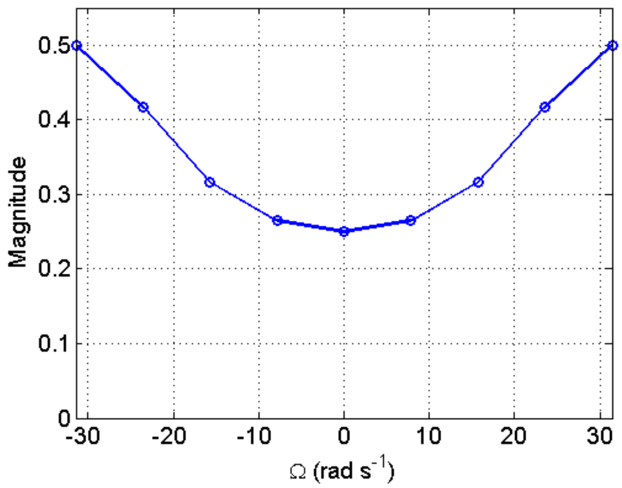

Figure 1: Frequency magnitude response $|H(e^{j\Omega T})|$ of the system.

- **Problem 2 [33 Points]**

1. An LTI system is described by the state-vector x and the state matrices

$$\mathbf{F} = \begin{bmatrix} -2 & 0 \\ -2 & 1 \end{bmatrix}, \; \mathbf{G} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \; \mathbf{H} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \; \mathbf{J} = 0.$$

Find the transformation $\mathbf{T}$ so that if $\mathbf{x} = \mathbf{Tz}$, the state matrices describing the system dynamics of z are in control canonical form. Compute the new matrices $\mathbf{A}, \mathbf{B}, \mathbf{C},$ and $\mathbf{D}$.

Such a $\mathbf{T}$ will be a 2×2 matrix, with second row t_2 of $\mathbf{T}^{-1}$ given by

$$t_2 = [0 \quad 1]\mathcal{C}^{-1}$$

where $\mathcal{C}$ is the controllability matrix, given by

$$\mathcal{C} = [\mathbf{G} \quad \mathbf{FG}] = \begin{bmatrix} 3 & -6 \\ 1 & -5 \end{bmatrix}.$$

$$\mathcal{C}^{-1} = \begin{bmatrix} 3 & -6 \\ 1 & -5 \end{bmatrix}^{-1} = \begin{bmatrix} 5/9 & -2/3 \\ 1/9 & -1/3 \end{bmatrix}.$$

$$t_2 = [0 \quad 1]\,\mathcal{C}^{-1} = [1/9 \quad -1/3].$$
$$t_1 = t_2\mathbf{F} = [4/9 \quad -1/3].$$

So,

$$\mathbf{T}^{-1} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 4/9 & -1/3 \\ 1/9 & -1/3 \end{bmatrix},$$

$$\Longrightarrow \mathbf{T} = \begin{bmatrix} 4/9 & -1/3 \\ 1/9 & -1/3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -3 \\ 1 & 4 \end{bmatrix}.$$

Now,

$$\mathbf{A} = \mathbf{T}^{-1}\mathbf{F}\mathbf{T} = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix},$$

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{G} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\mathbf{C} = \mathbf{H}\mathbf{T} = \begin{bmatrix} 1 & -4 \end{bmatrix},$$

$$\mathbf{D} = \mathbf{J} = 0.$$

2. Draw a block diagram of the system using a minimum number of integrators, gains and summers.

Through inspecting the matrices **A** and **C** of the control canonical form, we can write down the transfer function as

$$H(s) = \frac{s-4}{s^2+s-2}.$$

Using this transfer function, a block diagram can be drawn as shown in Fig. 2

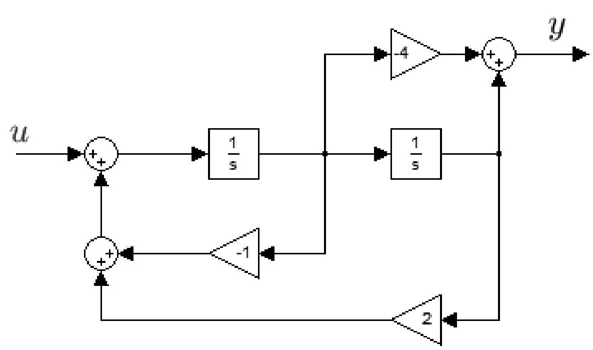

Figure 2: Block diagram of the system in Problem 2.

- **Problem 3 [33 Points]**
  Suppose that you have been tasked to implement a compensator

$$C(s) = 3\frac{s+1}{s+10},$$

on a digital system, whose A/D and D/A are clocked at 100 Hz. Please follow the following steps to obtain an approximation to the continuous-time controller.

1. Map individually, the pole -10 and the zero -1 in the s-plane to a pole β and a zero α in the z-plane by the relation $z = e^{sT}$.

   Using $T = 1/f_s = 1/100$, the pole at $s = -10$ maps on to

$$z = e^{-10/100} \simeq 0.905,$$

   and the zero at $s = -1$ maps on to

$$z = e^{-1/100} = 0.99.$$

   So, $\alpha = 0.99$ and $\beta = 0.905$.

2. Setup a discrete-time system

$$D(z) = K\frac{z-\alpha}{z-\beta},$$

   with K as yet unknown.

$$D(z) = K\frac{z-0.99}{z-0.905}.$$

3. Use the Final Value Theorem in continuous-time to compute the DC gain of $C(s)$.

   To emulate the DC input, we can use unit step as input, whose Laplace transform is $\frac{1}{s}$. So,

$$\text{DC gain} = \lim_{s\to 0} sC(s)\frac{1}{s} = \lim_{s\to 0} C(s) = 0.3.$$

4. Invoke the Final Value Theorem in discrete-time to get an expression for the DC gain of $D(z)$ in terms of K.

   To emulate the DC input, we can use unit step as input, whose z-transform is $\frac{z}{z-1}$. So,

$$\text{DC gain} = \lim_{z\to 1}(z-1)D(z)\frac{z}{z-1} = \lim_{z\to 1} zD(z) = 0.1053K.$$

5. Equate the two DC gains to obtain a value of K.

$$0.1053K = 0.3 \implies K = 2.85.$$

6. Convert the discrete-time controller into a difference equation.

$$D(z) = \frac{U(z)}{E(z)} = 2.85\frac{z-0.99}{z-0.905}$$
$$U(z)(z-0.905) = 2.85E(z)(z-0.99)$$
$$zU(z) - 0.905U(z) = 2.85zE(z) - 2.82E(z),$$

   and now taking inverse z-transform, we get

$$u_{k+1} - 0.905u_k = 2.85e_{k+1} - 2.82e_k,$$
$$\implies u_k - 0.905u_{k-1} = 2.85e_k - 2.82e_{k-1}.$$

7. Write a pseudo-code for this controller.

```
% initialize parameters and states
T = 0.01;  u = 0;
while(1)
    % read current input values of 'e' at A/D
    e =  ;
    u = 0.905 * u + 2.85 * e - 2.82 * e_{-1};
    e_{-1} = e;
    % wait until the next time interval
    sleep(T);
end while
```

8. Setup a block diagram of the controller using delay elements, gains and summers.

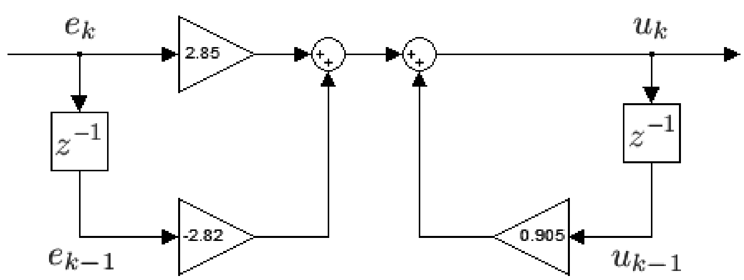

Figure 3: Block diagram of the discretized system.

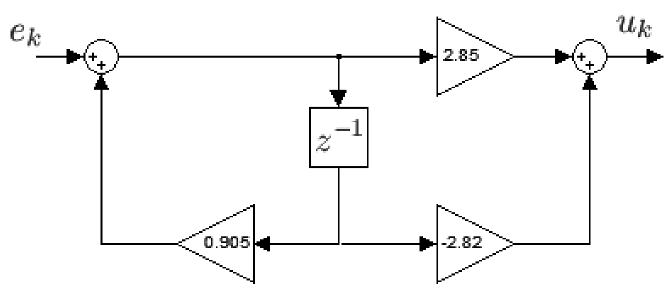

Figure 4: A simplified alternate block diagram of the discretized system.

The procedure followed is known as the Matched Pole-Zero (MPZ) method for digitizing of controllers.