EE 561: Digital Control Systems

Quiz 1: Solution

Marks: 100 *Friday, September* 27*, Fall* 2013.

- **Problem 1**

1. Find the z-transform of $f_k = ka^k$. You may use the property

$$\mathcal{Z}\{h_k a^k\} = H\left(\frac{z}{a}\right).$$

$$\mathcal{Z}\{\mathbb{1}_k\} = \sum_{k=-\infty}^{\infty} \mathbb{1}_k z^{-k} = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}.$$

$$\frac{d}{dz}\mathcal{Z}\{\mathbb{1}_k\} = \frac{d}{dz}\sum_{k=0}^{\infty} z^{-k} = \sum_{k=0}^{\infty}\frac{d}{dz}z^{-k} = \sum_{k=0}^{\infty}(-k)z^{-k-1} = -z^{-1}\sum_{k=0}^{\infty} kz^{-k} = -z^{-1}\mathcal{Z}\{k\},$$

$$\Rightarrow \mathcal{Z}\{k\} = -z\frac{d}{dz}\mathcal{Z}\{\mathbb{1}_k\} = -z\frac{d}{dz}\left(\frac{z}{z-1}\right) = -z\left(\frac{-1}{(z-1)^2}\right) = \frac{z}{(z-1)^2}.$$

Now,

$$\mathcal{Z}\{f_k\} = \mathcal{Z}\{ka^k\} = \frac{\frac{z}{a}}{(\frac{z}{a}-1)^2} = \frac{az}{(z-a)^2},$$

from the observation that

$$\sum_{k=0}^{\infty} k\left(\frac{z}{a}\right)^{-k} = \sum_{k=0}^{\infty} ka^k z^{-k}.$$

2. Solve and plot

$$y_n - 6y_{n-1} + 9y_{n-2} = u_n, \quad n = 0, 1, 2, ...$$

with initial conditions $y_{-1} = 1$, $y_{-2} = 0$ and input $u_n = 0$ for all n. Use the z-transform method. Verify your answer for $n = 0, 1$ and 2, by solving the equation through ordinary recursions.

$$\mathcal{Z}\{y_n\} = \sum_{n=0}^{\infty} y_n z^{-n} = Y(z).$$

$$\mathcal{Z}\{y_{n-1}\} = \sum_{n=0}^{\infty} y_{n-1}z^{-n} = z^{-1}\sum_{n=0}^{\infty} y_{n-1}z^{-(n-1)} = z^{-1}\left(y_{-1}z + \sum_{n=1}^{\infty} y_{n-1}z^{-(n-1)}\right)$$

$$= z^{-1}\left(y_{-1}z + \sum_{n=0}^{\infty} y_n z^{-n}\right) = y_{-1} + z^{-1}\sum_{n=0}^{\infty} y_n z^{-n} = y_{-1} + z^{-1}Y(z).$$

$$\mathcal{Z}\{y_{n-2}\} = \sum_{n=0}^{\infty} y_{n-2}z^{-n} = z^{-2}\sum_{n=0}^{\infty} y_{n-2}z^{-(n-2)} = z^{-2}\left(y_{-2}z^2 + y_{-1}z + \sum_{n=2}^{\infty} y_{n-2}z^{-(n-2)}\right)$$

$$= z^{-2}\left(y_{-2}z^2 + y_{-1}z + \sum_{n=0}^{\infty} y_n z^{-n}\right) = y_{-2} + y_{-1}z^{-1} + z^{-2}\sum_{n=0}^{\infty} y_n z^{-n}$$

$$= y_{-2} + z^{-1}y_{-1} + z^{-2}Y(z).$$

$$\mathcal{Z}\{u_n\} = \mathcal{Z}\{0\} = 0.$$

Plugging in z-transforms and initial conditions, we obtain

$$Y(z) - 6(1 + z^{-1}Y(z)) + 9(0 + z^{-1} + z^{-2}Y(z)) = 0,$$

$$\Rightarrow Y(z)(z^2 - 6z + 9) = 6z^2 - 9z,$$

$$\Rightarrow Y(z) = \frac{6z^2 - 9z}{z^2 - 6z + 9} = \frac{6z^2 - 9z}{(z-3)^2} = \frac{6z}{z-3} + \frac{9z}{(z-3)^2}.$$

Taking inverse z-transform of $Y(z)$,

$$\mathcal{Z}^{-1}\{Y(z)\} = y_n = 6\mathcal{Z}^{-1}\left\{\frac{z}{z-3}\right\} + 3\mathcal{Z}^{-1}\left\{\frac{3z}{(z-3)^2}\right\}.$$

Now using the results from part 1,

$$y_n = 6(3^n) + 3(n3^n) = (6+3n)3^n.$$

From this closed form of y_n,

$$\begin{aligned}
y_0 &= (6+3(0))3^0 = 6,\\
y_1 &= (6+3(1))3 = 27,\\
y_2 &= (6+3(2))3^2 = 108,\\
y_3 &= (6+3(3))3^3 = 405,\\
y_4 &= (6+3(4))3^4 = 1458,\\
y_5 &= (6+3(5))3^5 = 5103.
\end{aligned}$$

By using the recursion

$$y_n = 6y_{n-1} - 9y_{n-2},$$

we obtain

$$\begin{aligned}
y_0 &= 6y_{-1} - 9y_{-2} = 6,\\
y_1 &= 6y_0 - 9y_{-1} = 6(6) - 9 = 27,\\
y_2 &= 6y_1 - 9y_0 = 6(27) - 9(6) = 162 - 54 = 108,
\end{aligned}$$

which verify the required results obtained using the closed form of y_n.

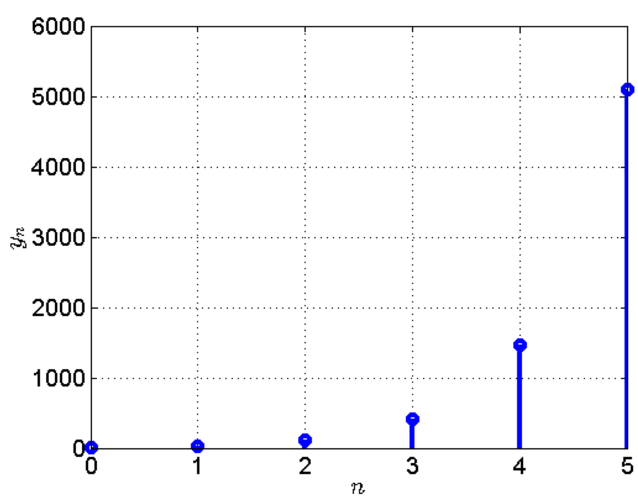

Figure 1: Plot of y_n.

3. Suggest an internal block diagram of this system using only delay elements, summers, and gains.

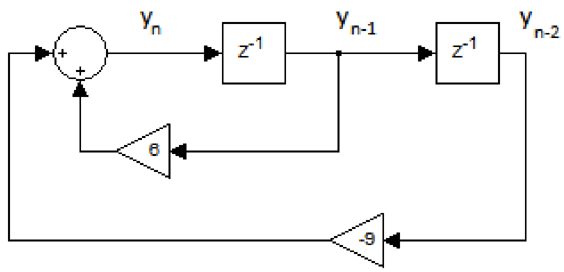

Figure 2: Block diagram for the system $y_n = 6y_{n-1} - 9y_{n-2}$.

4. Compute and plot the impulse response of the system (by any method).
For impulse response h_k, $y_n = 0$ for $n < 0$,

$$\frac{Y(z)}{U(z)} = H(z) = \frac{1}{1 - 6z^{-1} + 9z^{-2}} = \frac{z^2}{(z-3)^2} = 3^{-1}z\left(\frac{3z}{(z-3)^2}\right).$$

Now,

$$h_n = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{3^{-1}z\left(\frac{3z}{(z-3)^2}\right)\right\},$$

and using the inverse z-transform of $\dfrac{az}{(z-a)^2}$ from part 1,

$$h_n = 3^{-1}(n+1)3^{n+1} = (n+1)3^n.$$

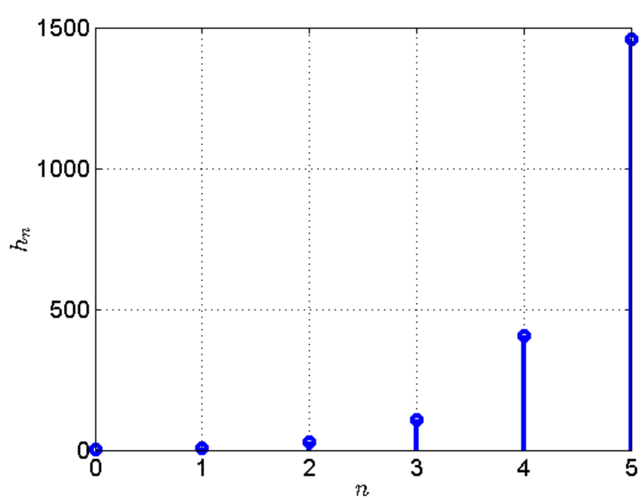

Figure 3: Plot of impulse response h_n.