**EE 561: Digital Control Systems**

# Problem Set 6 Solution

*Due on Dec 6 at 5PM* *Fall 2013*

- **Problem 1 [15 Points]**

The state-matrices in Control Canonical form for a system are given by

$$\mathbf{\Phi}_c = \begin{bmatrix} 6 & -13 & 10 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ \mathbf{\Gamma}_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{C}_c = \begin{bmatrix} 1 & -4 & 3 \end{bmatrix}, \ \mathbf{D}_c = 0.$$

Convert these matrices into Modal Canonical form having the new matrices $\mathbf{\Phi}_m, \mathbf{\Gamma}_m, \mathbf{C}_m$ and $\mathbf{D}_m$.

**Method 1:**

Eigenvalues of $\mathbf{\Phi}_c$ are $2, 2+j$ and $2-j$. Their respective eigenvectors are

$$\begin{bmatrix} 0.8792 \\ 0.4364 \\ 0.2182 \end{bmatrix}, \begin{bmatrix} 0.8980 \\ 0.3592 - 0.1796j \\ 0.1078 - 0.1437j \end{bmatrix} \text{ and } \begin{bmatrix} 0.8980 \\ 0.3592 + 0.1796j \\ 0.1078 + 0.1437j \end{bmatrix}.$$

Forming first column of the transformation matrix $\mathbf{T}$ from the real eigenvector, and rest of the two columns of $\mathbf{T}$ from real and imaginary parts of complex conjugate eigenvectors,

$$\mathbf{T} = \begin{bmatrix} 0.8792 & 0.8980 & 0 \\ 0.4364 & 0.3592 & 0.1796 \\ 0.2182 & 0.1078 & 0.1437 \end{bmatrix}$$

.

$$\mathbf{\Phi}_m = \mathbf{T}^{-1}\mathbf{\Phi_c}\mathbf{T} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix},$$

$$\mathbf{\Gamma}_m = \mathbf{T}^{-1}\mathbf{\Gamma_c} = \begin{bmatrix} 4.4572 \\ -3.2503 \\ -4.3297 \end{bmatrix},$$

$$\mathbf{C}_m = \mathbf{C}_c\mathbf{T} = [-0.2118 \quad -0.2154 \quad -0.2873],$$

$$\mathbf{D}_m = 0.$$

**Method 2:**

$$G(z) = \frac{z^2 - 4z + 3}{z^3 - 6z^2 + 13z - 10} = -\frac{1}{z-2} + \frac{2z-4}{z^2 - 4z + 5},$$

because $G(z)$ has a pair of non-real valued poles. In the canonical form, we can write the blocks corresponding to complex eigenvalues in Control Canonical form, and the cells corresponding to real and distinct eigenvalues in original Modal Canonical form as

$$\mathbf{\Phi}_m = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & -5 \\ 0 & 1 & 0 \end{bmatrix}, \ \mathbf{\Gamma}_m = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{C}_m = \begin{bmatrix} -1 & 2 & -4 \end{bmatrix}, \ \mathbf{D}_m = 0.$$

- **Problem 2 [20 Points]**

Find out the state $\mathbf{x}^* = [x_1^* \ \ x_2^*]^\top$ and the input $\mathbf{u}^* = [u_1^* \ \ u_2^*]^\top$ that minimize the cost

$$J(x_1, x_2, u_1, u_2) = 2x_1^2 + 2x_1x_2 + 3x_2^2 + 5u_1^2 + 3u_1u_2 + 7u_2^2,$$

to drive a system that is subject to the constraints

$$x_1 = 1 - x_2,$$
$$u_1 = 1 + u_2.$$

The only stationary point of $J(x_1, x_2, u_1, u_2)$ is its one global minimum. Why there are no other stationary points such as local minima/maxima, global maxima or saddle points?

Let's define a new cost function

$$\bar{J} = 2x_1^2 + 2x_1x_2 + 3x_2^2 + 5u_1^2 + 3u_1u_2 + 7u_2^2 + \lambda_1(x_1 + x_2 - 1) + \lambda_2(u_1 - u_2 - 1).$$

At the minimum of the cost function,

$$\begin{aligned}
\frac{\partial \bar{J}}{\partial x_1} &= 4x_1^* + 2x_2^* + \lambda_1 = 0, \\
\frac{\partial \bar{J}}{\partial x_2} &= 2x_1^* + 6x_2^* + \lambda_1 = 0, \\
\frac{\partial \bar{J}}{\partial u_1} &= 10u_1^* + 3u_2^* + \lambda_2 = 0, \\
\frac{\partial \bar{J}}{\partial u_2} &= 3u_1^* + 14u_2^* - \lambda_2 = 0, \\
\frac{\partial \bar{J}}{\partial \lambda_1} &= x_1^* + x_2^* - 1 = 0, \\
\frac{\partial \bar{J}}{\partial \lambda_2} &= u_1^* - u_2^* - 1 = 0.
\end{aligned}$$

$$\begin{bmatrix} x_1^* \\ x_2^* \\ u_1^* \\ u_2^* \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 & 0 & 1 & 0 \\ 2 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 10 & 3 & 0 & 1 \\ 0 & 0 & 3 & 14 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 17/30 \\ -13/30 \\ -10/3 \\ -131/30 \end{bmatrix}.$$

- **Problem 3 [30 Points]**

  For a plant with state $\mathbf{x}(t)$ and state matrices

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ -3 & -5 \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{H} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ \mathbf{J} = 0,$$

  (a) Design a controller, without using integral control, to yield a 5% overshoot and a settling time of 1 s.

  With the given design specifications, $\zeta \simeq 0.5$ and $\omega_n \simeq 9.2$, which leads to the characteristic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 9.2s + 84.64.$$

  Using Ackermann,

$$\mathbf{K} = [0 \ \ 1]\mathcal{C}^{-1}\alpha(\mathbf{F}) = [81.64 \ \ 4.2].$$

  (b) Find out the steady-state error for a unit-step input, with this controller.

  At steady state, $\dot{\mathbf{x}} = 0$, therefore

$$y_{ss} = -\mathbf{H}\mathbf{x}_{ss} + \mathbf{J}u = -\mathbf{H}(\mathbf{F} - \mathbf{G}\mathbf{K})^{-1}\mathbf{G} = 0.0118.$$

  Steady-state error

$$e_{ss} = r - y_{ss} = 1 - 0.0118 = 0.9882.$$

  (c) Now design the controller using integral control.

  For continuous-time integral control, the control input $u$ will be given by

$$u(t) = k_i \int_0^t (r - y)d\tau - \mathbf{K}\mathbf{x}(t).$$

  Let a new state $x_i = \int_0^t (r - y)d\tau$, which leads to

$$u(t) = k_i x_i - \mathbf{K}\mathbf{x},$$

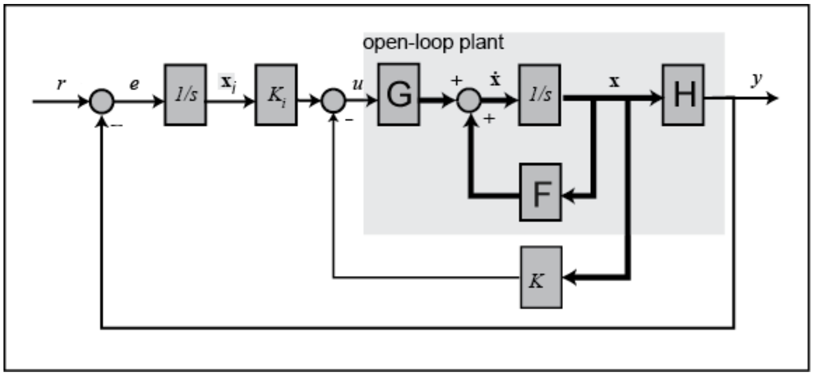


Figure 1: Block diagram for continuous time integral control.

and

$$\dot{x}_i = r - y = r - \mathbf{H}\mathbf{x}.$$

Let the new state vector $\tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ x_i \end{bmatrix}$. Now,

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u = (\mathbf{F} - \mathbf{G}\mathbf{K})\mathbf{x} + \mathbf{G}k_i x_i = [\mathbf{F} - \mathbf{G}\mathbf{K} \;\; \mathbf{G}k_i] \begin{bmatrix} \mathbf{x} \\ x_i \end{bmatrix},$$

$$\dot{x}_i = [-\mathbf{H} \;\; 0] \begin{bmatrix} \mathbf{x} \\ x_i \end{bmatrix} + r.$$

Combining these into one single system,

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{x}_i \end{bmatrix} = \dot{\tilde{\mathbf{x}}} = \begin{bmatrix} \mathbf{F} - \mathbf{G}\mathbf{K} & \mathbf{G}k_i \\ -\mathbf{H} & 0 \end{bmatrix} \tilde{\mathbf{x}} + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r,$$

$$y = [\mathbf{H} \;\; 0]\tilde{\mathbf{x}}.$$

From this system, $\mathbf{F}_{\text{new}} = \begin{bmatrix} \mathbf{F} - \mathbf{G}\mathbf{K} & \mathbf{G}k_i \\ -\mathbf{H} & 0 \end{bmatrix}$, $\mathbf{G}_{\text{new}} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$, $\mathbf{H}_{\text{new}} = \begin{bmatrix} \mathbf{H} & 0 \end{bmatrix}$, and we can find the characteristic equation from

$$\det(s\mathbf{I} - \mathbf{F}_{\text{new}}) = s^3 + (5 + k_2)s^2 + (3 + k_1)s + k_i.$$

But the desired characteristic equation is $s^2 + 9.2s + 84.64$. To make it comparable to the characteristic equation, we need to add a root to $s^2 + 9.2s + 84.64$ such that the new root is sufficiently far to the left of its current roots. By choosing $s = -50$, we have the new desired characteristic equation as

$$(s + 50)(s^2 + 9.2s + 84.64) = s^3 + 59.2s^2 + 544.6s + 4232.$$

Comparing the coefficients, we obtain

$$k_1 = 541.6, k_2 = 54.2, k_i = 4232.$$

(d) Find out the steady state error with this integral control, for a unit-step input. What 'type' of system is this?

$$y_{ss} = -\mathbf{H}_{\text{new}}(\mathbf{F}_{\text{new}} - \mathbf{G}_{\text{new}}\mathbf{K}_{\text{new}})^{-1}\mathbf{G}_{\text{new}} = 1.$$

Steady-state error

$$e_{ss} = r - y_{ss} = 1 - 1 = 0.$$

Zero stead-state error for a step-input, so the system is Type I.

- **Problem 4 [35 Points]**
  The transfer function for a system is given by

$$G(z) = \frac{z + 0.967}{z^2 - 1.67z + 0.91}.$$

(a) Find control feedback $\mathbf{K}$ and observer gain $\mathbf{L}$ that place control poles at $z = 0.8 \pm j0.2$ and observer poles at $z = 0.6 \pm j0.3$.

Considering the system in Control canonical form, we have

$$\mathbf{\Phi}_c = \begin{bmatrix} 1.67 & -0.91 \\ 1 & 0 \end{bmatrix}, \ \mathbf{\Gamma}_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{C}_c = [1 \ \ 0.967], \ \mathbf{D}_c = 0.$$

From the design poles, the characteristic polynomial can be found as

$$\alpha(z) = (z - 0.8 + 0.2j)(z - 0.8 - 0.2j) = z^2 - 1.6z + 0.68.$$

It follows that

$$\mathbf{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 - a_1 \\ \alpha_2 - a_2 \end{bmatrix} = \begin{bmatrix} 0.07 \\ -0.23 \end{bmatrix}.$$

From the design poles of observer, the characteristic polynomial can be found as

$$\beta(z) = (z - 0.6 + 0.3j)(z - 0.6 - 0.3j) = z^2 - 1.2z + 0.45.$$

Using Ackermann's,

$$\mathbf{L} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \beta(\mathbf{\Phi}_c)\mathcal{O}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.2144 \\ 0.2643 \end{bmatrix}$$

(b) Draw the block-diagram of the system in state-command structure.

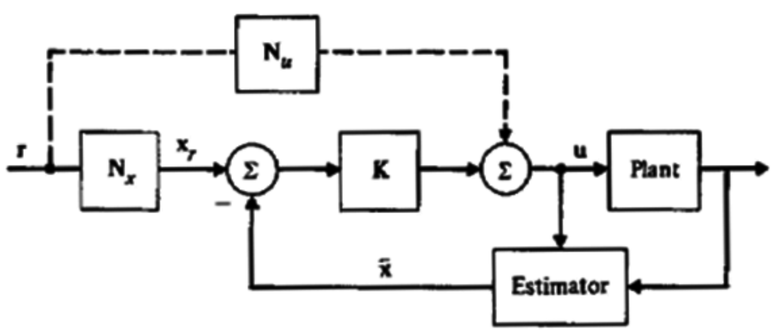


Figure 2: State-command structure with $\mathbf{N}_x$ and $\mathbf{N}_u$.

(c) For the state-command structure, find out $\mathbf{N}_x$ and $\mathbf{N}_u$, and plot $y(k)$ for a unit step input.

$$\begin{bmatrix} \mathbf{N}_x \\ \mathbf{N}_u \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_c - \mathbf{I} & \mathbf{\Gamma_c} \\ \mathbf{C}_c & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5084 \\ 0.5084 \\ \cdots\cdots \\ 0.1220 \end{bmatrix}.$$

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{\Phi}_c\mathbf{x}(k) + \mathbf{\Gamma}_c u(k) = \mathbf{\Phi}_c\mathbf{x}(k) + \mathbf{\Gamma}_c(\mathbf{N}_u + \mathbf{K}\mathbf{N}_\mathbf{x})r(k) - \mathbf{\Gamma}_c\mathbf{K}\hat{\mathbf{x}}(k) \\ &= \mathbf{\Phi}_c\mathbf{x}(k) + \mathbf{\Gamma}_c(\mathbf{N}_u + \mathbf{K}\mathbf{N}_\mathbf{x})r(k) - \mathbf{\Gamma}_c\mathbf{K}(\mathbf{x}(k) - \mathbf{e}_\mathbf{x}(k)) \\ &= (\mathbf{\Phi}_c - \mathbf{\Gamma}_c\mathbf{K})\mathbf{x}(k) + \mathbf{\Gamma}_c\mathbf{K}\mathbf{e}_\mathbf{x}(k) + \mathbf{\Gamma}_c(\mathbf{N}_u + \mathbf{K}\mathbf{N}_\mathbf{x})r(k). \end{aligned}$$

Similarly,

$$\begin{aligned} \hat{\mathbf{x}}(k+1) &= \mathbf{\Phi}_c\hat{\mathbf{x}}(k) + \mathbf{\Gamma}_c u(k) + \mathbf{L}(y - \hat{y}) \\ &= \mathbf{\Phi}_c\hat{\mathbf{x}}(k) + \mathbf{\Gamma}_c(\mathbf{N}_u + \mathbf{K}\mathbf{N}_\mathbf{x})r(k) - \mathbf{\Gamma}_c\mathbf{K}\hat{\mathbf{x}}(k) + \mathbf{L}(\mathbf{C}_c\mathbf{x}(k) - \mathbf{C}_c\hat{\mathbf{x}}(k)) \\ &= \mathbf{L}\mathbf{C}_c\mathbf{x}(k) + (\mathbf{\Phi}_c - \mathbf{\Gamma}_c\mathbf{K} - \mathbf{L}\mathbf{C}_c)\hat{\mathbf{x}}(k) + \mathbf{\Gamma}_c(\mathbf{N}_u + \mathbf{K}\mathbf{N}_\mathbf{x})r(k). \end{aligned}$$

Let $\mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1) = \mathbf{e}_{\mathbf{x}}(k+1)$.

$$\begin{aligned}\mathbf{e}_{\mathbf{x}}(k+1) &= (\mathbf{\Phi}_c - \mathbf{\Gamma}_c\mathbf{K})\mathbf{e}_{\mathbf{x}}(k) - \mathbf{L}\mathbf{C}_c\mathbf{e}_{\mathbf{x}}(k) + \mathbf{\Gamma}_c\mathbf{K}\mathbf{e}_{\mathbf{x}}(k) \\ &= (\mathbf{\Phi}_c - \mathbf{L}\mathbf{C}_c)\mathbf{e}_{\mathbf{x}}(k).\end{aligned}$$

The system with states and error states can be coupled as

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{e}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_c - \mathbf{\Gamma}_c\mathbf{K} & \mathbf{\Gamma}_c\mathbf{K} \\ \mathbf{0} & \mathbf{\Phi}_c - \mathbf{L}\mathbf{C}_c \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{e}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{\Gamma}_c(\mathbf{N}_u + \mathbf{K}\mathbf{N}_{\mathbf{x}}) \\ \mathbf{0} \end{bmatrix} r(k),$$

$$y(k) = [\mathbf{C}_c \ \ \mathbf{0}] \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{e}(k) \end{bmatrix}.$$

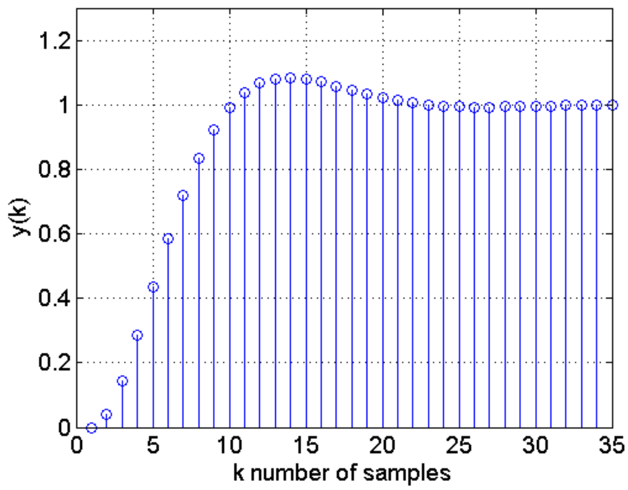


Figure 3: Step response in which zero steady-state error can be observed.

(d) With $\bar{\mathbf{N}} = \mathbf{K}\mathbf{N}_x + \mathbf{N}_u$, draw the new block diagram substituting $\bar{\mathbf{N}}$ for $\mathbf{N}_x$ and $\mathbf{N}_u$.

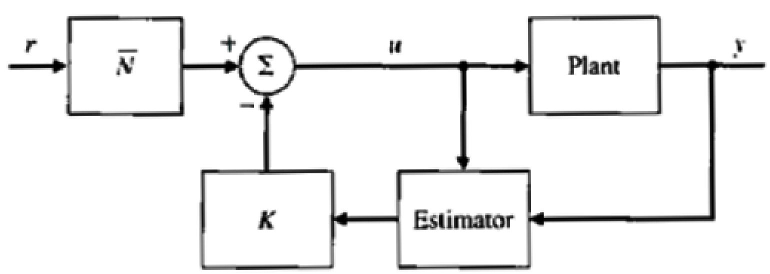


Figure 4: State-command structure with $\bar{\mathbf{N}} = \mathbf{N}_u + \mathbf{K}\mathbf{N}_{\mathbf{x}}$.