EE 561: Digital Control Systems

Problem Set 5

Due on Nov. 25 in class *Fall 2013*

- **Problem 1 [15 Points]**

For this problem, refer to section 6.3.2 of 'Digital Control of Dynamic Systems' by Franklin et al.. A continuous-time system with state $\mathbf{x}$ is described by the state matrices

$$\mathbf{F} = \begin{bmatrix} -2 & -1 & -3 \\ 0 & -2 & 1 \\ -7 & -8 & -9 \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \ \mathbf{H} = \begin{bmatrix} 4 & 6 & 8 \end{bmatrix}, \ \mathbf{J} = 0.$$

(a) Find out the transfer function $H(s)$ for the system.

Finding out the transfer function using the expression $H(s) = \mathbf{H}(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G} + \mathbf{J}$ will be a hectic task. So, alternatively, we can convert the matrices into Control Canonical form and directly write the transfer function from it.

$$\mathcal{C}^{-1} = \left[\mathbf{G} \ \ \mathbf{FG} \ \ \mathbf{F}^2\mathbf{G}\right]^{-1} = \begin{bmatrix} 2 & -11 & 142 \\ 1 & 0 & -40 \\ 2 & -40 & 437 \end{bmatrix}^{-1} = \begin{bmatrix} 0.5011 & 0.2734 & -0.1378 \\ 0.1619 & -0.1848 & -0.0695 \\ 0.0125 & -0.0182 & -0.0034 \end{bmatrix}.$$

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{t}_3\mathbf{F}^2 \\ \mathbf{t}_3\mathbf{F} \\ [0 \ \ 0 \ \ 1]\,\mathcal{C}^{-1} \end{bmatrix} = \begin{bmatrix} 0.1751 & 0.0961 & 0.2769 \\ -0.0009 & 0.0514 & -0.0247 \\ 0.0125 & -0.0182 & -0.0034 \end{bmatrix},$$

$$\implies \mathbf{T} = \begin{bmatrix} 2 & 15 & 53 \\ 1 & 13 & -13 \\ 2 & -14 & -29 \end{bmatrix}.$$

$$\mathbf{A}_c = \mathbf{T}^{-1}\mathbf{F}\mathbf{T} = \begin{bmatrix} -13 & -27 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\mathbf{C}_c = \mathbf{H}\mathbf{T} = [30 \ \ 26 \ \ -98].$$

$$H(s) = \frac{30s^2 + 26s - 98}{s^3 + 13s^2 + 27s + 3}.$$

(b) Discretize the system using a First-order Hold, and find out the new overall transfer function $G(z)$ for $T = 1$.

$$G(z) = \frac{(z-1)^2}{z}\mathcal{Z}\left\{\frac{H(s)}{s^2}\right\}.$$

$$\mathcal{Z}\left\{\frac{H(s)}{s^2}\right\} = \mathcal{Z}\left\{\frac{30s^2 + 26s - 98}{s^5 + 13s^4 + 27s^3 + 3s^2}\right\}.$$

Using the Matlab command `c2d(sys,1,'impulse')`,

$$\mathcal{Z}\left\{\frac{H(s)}{s^2}\right\} = \frac{0.7854z^4 - 4.309z^3 + 0.1872z^2 + 0.02493z}{z^5 - 2.976z^4 + 3.03z^3 + -1.131z^2 + 0.07747z}$$

$$= \frac{0.7854z^3 - 4.309z^2 + 0.1872z + 0.02493}{z^4 - 2.976z^3 + 3.03z^2 - 1.131z + 0.07747}.$$

$$G(z) = \frac{(z-1)^2}{z}\left(\frac{0.7854z^3 - 4.309z^2 + 0.1872z + 0.02493}{z^4 - 2.976z^3 + 3.03z^2 + -1.131z + 0.07747}\right)$$

$$= \frac{0.7854z^5 - 5.88z^4 + 9.591z^3 - 4.659z^2 + 0.1374z + 0.02493}{z^5 - 2.976z^4 + 3.03z^3 - 1.131z^2 + 0.07747z}.$$

(c) Now find out the new state matrices $\mathbf{\Phi}, \mathbf{\Gamma}, \mathbf{C_D}$ and $\mathbf{D_D}$. You can verify your result using `sysd = c2d(sys,T,'foh')` command on Matlab.

Using Control Canonical form,

$$\mathbf{\Phi} = \begin{bmatrix} 2.976 & 3.03 & 1.131 & -0.07747 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{\Gamma} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{aligned} \mathbf{C}_d &= \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & b_3 - a_3 b_0 & b_4 - a_4 b_0 & b_5 - a_5 b_0 \end{bmatrix} \\ &= \begin{bmatrix} -3.5426 & 7.2113 & -3.7707 & 0.0766 & 0.0244 \end{bmatrix}, \\ \mathbf{D}_d &= b_0 = 0.7854. \end{aligned}$$

- **Problem 2 [15 Points]**

A continuous-time system with state $\mathbf{x}$ is described by the state matrices

$$\mathbf{F} = \begin{bmatrix} -2 & 0 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{J} = 0,$$

Discretize the system using Tustin's approximation, and find the new matrices $\mathbf{\Phi}, \mathbf{\Gamma}, \mathbf{C}$ and $\mathbf{D}$ for the new system

$$\begin{aligned} \mathbf{w}(k+1) &= \mathbf{\Phi}\mathbf{w}(k) + \mathbf{\Gamma}u(k) \\ y(k) &= \mathbf{C}\mathbf{w}(k) + \mathbf{D}u(k), \end{aligned}$$

with the new state variable $\mathbf{w}(k) = \frac{1}{\sqrt{T}}\left(\mathbf{I} - \frac{T}{2}\mathbf{F}\right)\mathbf{x}(k) - \frac{\sqrt{T}}{2}\mathbf{G}u(k)$. You can verify your result using `sysd = c2d(sys,T,'tustin')` command on Matlab.

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{F}\mathbf{x} + \mathbf{G}u, \\ s\mathbf{X}(s) &= \mathbf{F}\mathbf{X}(s) + \mathbf{G}U(s) \end{aligned}$$

Replacing s by $\frac{2}{T}\frac{z-1}{z+1}$,

$$\begin{aligned} \frac{2}{T}\frac{z-1}{z+1}\mathbf{X}(z) &= \mathbf{F}\mathbf{X}(z) + \mathbf{G}U(z), \\ (z-1)\mathbf{X}(z) &= \frac{T}{2}(z+1)\left(\mathbf{F}\mathbf{X}(z) + \mathbf{G}U(z)\right), \\ z\mathbf{X}(z) - \mathbf{X}(z) &= \frac{T}{2}\mathbf{F}z\mathbf{X}(z) + \frac{T}{2}\mathbf{G}zU(z) + \frac{T}{2}\mathbf{F}\mathbf{X}(z) + \frac{T}{2}\mathbf{G}U(z) \end{aligned}$$

$$\begin{aligned} z\mathbf{X}(z) - \frac{T}{2}\mathbf{F}z\mathbf{X}(z) - \frac{T}{2}\mathbf{G}zU(z) &= \mathbf{X}(z) + \frac{T}{2}\mathbf{F}\mathbf{X}(z) + \frac{T}{2}\mathbf{G}U(z), \\ \left(\mathbf{I} - \frac{T}{2}\mathbf{F}\right)z\mathbf{X}(z) - \frac{T}{2}\mathbf{G}zU(z) &= \left(\mathbf{I} + \frac{T}{2}\mathbf{F}\right)\mathbf{X}(z) + \frac{T}{2}\mathbf{G}U(z), \end{aligned}$$

Dividing both sides by $\sqrt{T}$ and taking inverse z-transform,

$$\frac{1}{\sqrt{T}}\left(\mathbf{I} - \frac{T}{2}\mathbf{F}\right)\mathbf{x}(k+1) - \frac{\sqrt{T}}{2}\mathbf{G}u(k+1) = \frac{1}{\sqrt{T}}\left(\mathbf{I} + \frac{T}{2}\mathbf{F}\right)\mathbf{x}(k) + \frac{\sqrt{T}}{2}\mathbf{G}u(k),$$

Substituting $\mathbf{w}(k+1)$ for the LHS, and $\sqrt{T}\left(\mathbf{I}-\frac{T}{2}\mathbf{F}\right)^{-1}\left(\mathbf{w}(k)+\frac{\sqrt{T}}{2}\mathbf{G}u(k)\right)$ for $\mathbf{x}(k)$,

$$\mathbf{w}(k+1) = \frac{1}{\sqrt{T}}\left(\mathbf{I}+\frac{T}{2}\mathbf{F}\right)\sqrt{T}\left(\mathbf{I}-\frac{T}{2}\mathbf{F}\right)^{-1}\left(\mathbf{w}(k)+\frac{\sqrt{T}}{2}\mathbf{G}u(k)\right)+\frac{\sqrt{T}}{2}\mathbf{G}u(k),$$

$$\implies \mathbf{w}(k+1) = \left(\mathbf{I}+\frac{T}{2}\mathbf{F}\right)\left(\mathbf{I}-\frac{T}{2}\mathbf{F}\right)^{-1}\mathbf{w}(k)+\frac{\sqrt{T}}{2}\left(\mathbf{I}+\left(\mathbf{I}+\frac{T}{2}\mathbf{F}\right)\left(\mathbf{I}-\frac{T}{2}\mathbf{F}\right)^{-1}\right)\mathbf{G}u(k).$$

Hence,

$$\mathbf{\Phi} = \left(\mathbf{I}+\frac{T}{2}\mathbf{F}\right)\left(\mathbf{I}-\frac{T}{2}\mathbf{F}\right)^{-1} = \begin{bmatrix} \dfrac{1-T}{1+T} & 0 \\ \dfrac{T^2}{(1+T)(1-T/2)} & \dfrac{1+T/2}{1-T/2} \end{bmatrix},$$

and

$$\mathbf{\Gamma} = \frac{\sqrt{T}}{2}\left(\mathbf{I}+\left(\mathbf{I}+\frac{T}{2}\mathbf{F}\right)\left(\mathbf{I}-\frac{T}{2}\mathbf{F}\right)^{-1}\right)\mathbf{G} = \frac{\sqrt{T}}{2}\begin{bmatrix} \dfrac{6}{1+T} \\ \dfrac{8T+2}{(1+T)(1-T/2)} \end{bmatrix}.$$

For $y(k)$,

$$\begin{aligned} y(k) = \mathbf{H}\mathbf{x}(k)+\mathbf{J}u(k) &= \sqrt{T}\mathbf{H}\left(\mathbf{I}-\frac{T}{2}\mathbf{F}\right)^{-1}\left(\mathbf{w}(k)+\frac{\sqrt{T}}{2}\mathbf{G}u(k)\right) \\ &= \sqrt{T}\mathbf{H}\left(\mathbf{I}-\frac{T}{2}\mathbf{F}\right)^{-1}\mathbf{w}(k)+\frac{T}{2}\mathbf{H}\left(\mathbf{I}-\frac{T}{2}\mathbf{F}\right)^{-1}\mathbf{G}u(k). \end{aligned}$$

Hence,

$$\mathbf{C} = \sqrt{T}\mathbf{H}\left(\mathbf{I}-\frac{T}{2}\mathbf{F}\right)^{-1} = \sqrt{T}\begin{bmatrix} 1+T & 0 \end{bmatrix},$$

and

$$\mathbf{D} = \frac{T}{2}\mathbf{H}\left(\mathbf{I}-\frac{T}{2}\mathbf{F}\right)^{-1} = \frac{3T}{2(1+T)}.$$

- **Problem 3 [20 Points]**
  The dynamic behavior of water-level in an irrigation channel shown in Fig. 1 is modeled by the differential equation

$$\dot{y}(t) = c_{\text{in}} u_{\text{in}}(t-\tau) - c_{\text{out}} u_{\text{out}}(t), \tag{1}$$

  where $y(t)$ is the downstream end water level variation; $u_{\text{in}}(t)$ measures inflow of water; $u_{\text{out}}(t)$ measures outflow of water; c_{in} and c_{out} are static gains; τ is the time taken by the water to cover the distance from A to B. Use the parameters $c_{\text{in}} = 10, c_{\text{out}} = 0$ (gate is closed), $\tau = 300$.

  (a) For value of the sampling interval $T = 60$, discretize the system using Forward Euler.
  Discretizing the equation

$$\dot{y} = 10 u_{\text{in}}(t - 300),$$

  by considering $300 = 5T$, we get,

$$\dot{y}(kT) \simeq \frac{y(kT+T) - y(kT)}{T} = 10 u_{\text{in}}(kT - 5T),$$

  which can be written as

$$y(k+1) = y(k) + 10T u_{\text{in}}(k-5),$$

$$y(k+1) = y(k) + 600 u_{\text{in}}(k-5).$$

  (b) For this part, refer to section 4.3.4 of 'Digital Control of Dynamic Systems' by Franklin et al.. Convert the discretized system into state-space representation, by incorporating the delay τ, and find out the new matrices $\mathbf{\Phi}, \mathbf{\Gamma}, \mathbf{C_D}$ and $\mathbf{D_D}$.

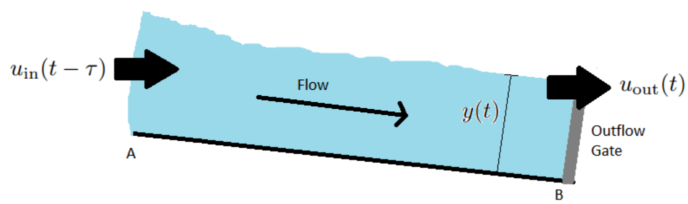

Figure 1: Irrigation channel model for Problem 3.

Let us declare $y(k) = x_1(k)$, which leads to

$$x_1(k+1) = x_1(k) + 600 u_{\text{in}}(k-5).$$

and declare new states as

$$\begin{aligned} u_{\text{in}}(k-5) &= x_2(k), \\ x_2(k+1) &= u_{\text{in}}(k-4) = x_3(k), \\ x_3(k+1) &= u_{\text{in}}(k-3) = x_4(k), \\ x_4(k+1) &= u_{\text{in}}(k-2) = x_5(k), \\ x_5(k+1) &= u_{\text{in}}(k-1) = x_6(k), \\ x_6(k+1) &= u_{\text{in}}(k) \end{aligned}$$

to form a new state vector $\mathbf{x} = [x_1 \;\; x_2 \;\; x_3 \;\; x_4 \;\; x_5 \;\; x_6]^\top$. So now, with the set of new difference equations,

$$\begin{aligned} x_1(k+1) &= x_1(k) + 600 x_2(k), \\ x_2(k+1) &= x_3(k), \\ x_3(k+1) &= x_4(k), \\ x_4(k+1) &= x_5(k), \\ x_5(k+1) &= x_6(k), \\ x_6(k+1) &= u_{\text{in}}(k), \end{aligned}$$

we can set up the state-space system as

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 600 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_{\text{in}}(k),$$

$$y(k) = [1 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0]\mathbf{x}(k).$$

- **Problem 4 [35 Points]**
  Consider the generalized transfer function

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + ... + b_n z^{-n}}{1 + a_1 z^{-1} + ... + a_n z^{-n}},$$

where $b_0 + b_1 z^{-1} + ... + b_n z^{-n} = b(z)$, and $1 + a_1 z^{-1} + ... + a_n z^{-n} = a(z)$.

(a) For the discrete-time system

$$\mathbf{x}(k+1) = \mathbf{\Phi_c x}(k) + \mathbf{\Gamma_c} u(k)$$
$$y(k) = \mathbf{C_c x}(k) + \mathbf{D_c} u(k),$$

derive the matrices $\mathbf{\Phi_c}, \mathbf{\Gamma_c}, \mathbf{C_c}$ and $\mathbf{D_c}$ for the Control Canonical form. You can use the states $X_i(z) = z^{-i}Q(z)$, where $Q(z) = \frac{U(z)}{a(z)} = \frac{Y(z)}{b(z)}$.

$$U(z) = a(z)Q(z) = Q(z) + a_1 z^{-1} Q(z) + a_2 z^{-2} Q(z) + \ldots + a_n z^{-n} Q(z), \tag{2}$$
$$\implies Q(z) = U(z) - a_1 z^{-1} Q(z) - a_2 z^{-2} Q(z) - \ldots - a_n z^{-n} Q(z),$$

Using the substitutions $Q(z) = zX_1(z)$ and $z^{-i}Q(z) = X_i(z)$, we have

$$zX_1(z) = -a_1 X_1(z) - a_2 X_2(z) - \ldots - a_n X_n(z) + U(z),$$
$$zX_2(z) = z^{-1}Q(z) = X_1(z),$$
$$\vdots$$
$$zX_n(z) = z^{-1}Q(z) = X_{n-1}(z).$$

With the z-transformed state-vector $\mathbf{X} = [X_1 \ \ X_2 \ \ \ldots \ \ X_n]^\top$, the system of equations can be written in the state-space form as

$$z\mathbf{X}(z) = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \ldots & -a_n \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & 0 \end{bmatrix} \mathbf{X}(z) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} U(z),$$

taking inverse z-transform,

$$\mathbf{x}(k+1) = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \ldots & -a_n \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k).$$

For finding out $y(k)$,

$$Y(z) = b(z)Q(z) = b_0 Q(z) + b_1 z^{-1} Q(z) + b_2 z^{-2} Q(z) + \ldots + b_n z^{-n} Q(z), \tag{3}$$

Applying (3) $-b_0$(2), we get

$$Y(z) - b_0 U(z) = (b_1 - a_1 b_0) z^{-1} Q(z) + (b_2 - a_2 b_0) z^{-2} Q(z) + \ldots + (b_n - a_n b_0) z^{-n} Q(z),$$

and substituting $z^{-i}Q(z) = X_i(z)$, leads to

$$Y(z) = (b_1 - a_1 b_0)X_1(z) + (b_2 - a_2 b_0)X_2(z) + \ldots + (b_n - a_n b_0)X_n(z) + b_0 U(z).$$

Taking inverse z-transform and converting the equation into state-space form yields,

$$y(k) = [b_1 - a_1 b_0 \quad b_2 - a_2 b_0 \quad \ldots \quad b_n - a_n b_0]\,\mathbf{x}(k) + b_0 u(k).$$

In the end,

$$\mathbf{\Phi}_c = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \ldots & -a_n \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{\Gamma}_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\mathbf{C}_c = [b_1 - a_1 b_0 \quad b_2 - a_2 b_0 \quad \ldots \quad b_n - a_n b_0], \quad \mathbf{D}_c = b_0.$$

(b) Draw the block diagram for the system in Control Canonical form.

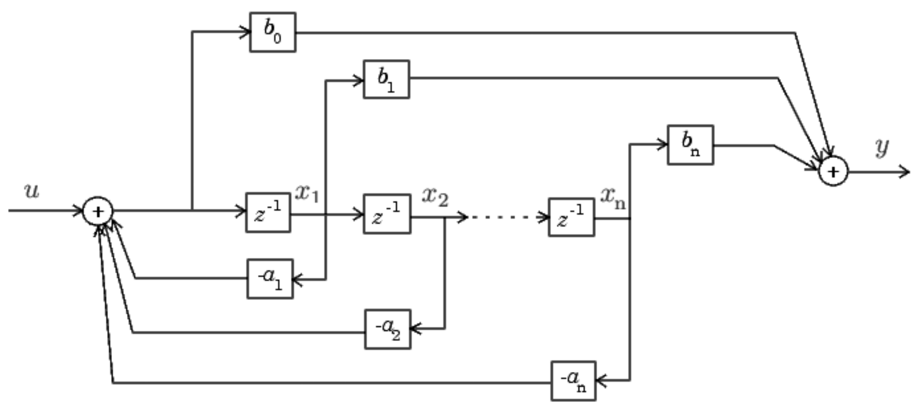

Figure 2: Block Diagram for Control Canonical form.

(c) Draw the dual of the block diagram in (b). [Hint: reverse the direction of all arrows, replace summers by junctions and junctions by summers. Note that the input and output need to be swapped.]

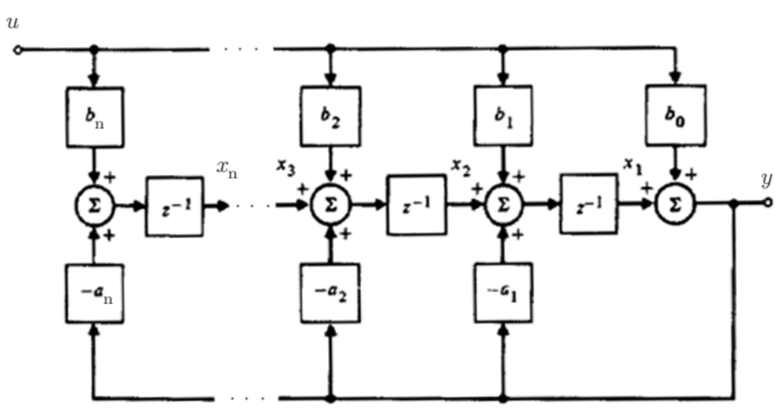

Figure 3: Block Diagram for Observer Canonical form.

(d) Using new states (outputs of unit delays) derive the matrices $\mathbf{\Phi_o}, \mathbf{\Gamma_o}, \mathbf{C_o}$ and $\mathbf{D_o}$ for the Observer Canonical form.

From Fig. 3, we can deduce that

$$y(k) = x_1(k) + b_0 u(k),$$

which implies that

$$y(k) = [1 \quad 0 \quad \dots \quad 0]\mathbf{x}(k) + b_0 u(k).$$

Analyzing the inputs of unit delays, we have

$$\begin{aligned}
zX_1(z) &= X_2(z) + b_1U(z) - a_1Y(z) = X_2(z) + b_1U(z) - a_1(X_1(z) + b_0U(z)) \\
&= -a_1X_1(z) + X_2(z) + (b_1 - a_1b_0)U(z), \\
zX_2(z) &= X_3(z) + b_2U(z) - a_2Y(z) = X_3(z) + b_2U(z) - a_2(X_1(z) + b_0U(z)) \\
&= -a_2X_1(z) + X_3(z) + (b_2 - a_2b_0)U(z), \\
&\vdots \\
zX_{n-1}(z) &= -a_{n-1}X_1(z) + X_n(z) + (b_{n-1} - a_{n-1}b_0)U(z), \\
zX(n) &= -a_nY(z) + b_nU(z).
\end{aligned}$$

Taking inverse z-transform of all these equations and converting into the state-space form, we get

$$\mathbf{x}(k+1) = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -a_{n-1} & 0 & \dots & 0 & 1 \\ -a_n & 0 & \dots & 0 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} b_1 - a_1b_0 \\ b_2 - a_2b_0 \\ b_3 - a_3b_0 \\ \vdots \\ b_n - a_nb_0 \end{bmatrix} u(k).$$

Finally,

$$\mathbf{\Phi}_o = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -a_{n-1} & 0 & \dots & 0 & 1 \\ -a_n & 0 & \dots & 0 & 0 \end{bmatrix}, \quad \mathbf{\Gamma}_o = \begin{bmatrix} b_1 - a_1b_0 \\ b_2 - a_2b_0 \\ b_3 - a_3b_0 \\ \vdots \\ b_n - a_nb_0 \end{bmatrix},$$

$$\mathbf{C}_o = [1 \quad 0 \quad \dots \quad 0], \quad \mathbf{D}_o = b_0.$$

(e) Now using the states $X_i(z) = \frac{1}{z-p_i}U(z)$, derive the matrices $\mathbf{\Phi_M}, \mathbf{\Gamma_M}, \mathbf{C_M}$ and $\mathbf{D_M}$ for Modal Canonical form in terms of c_i and p_i, where

$$c_i = \lim_{z \to p_i} \left[\frac{Y(z)}{U(z)}(z - p_i) \right]$$

and p_i are the roots of the equation $a(z) = 0$.

For this problem, you can seek guidance from section 4.2.3 of 'Digital Control of Dynamic Systems' by Franklin et al..

$H(z)$ can be broken into partial fractions as

$$H(z) = \frac{Y(z)}{U(z)} = \frac{c_1}{z - p_1} + \dots + \frac{c_n}{z - p_n},$$

$$\implies Y(z) = \frac{c_1}{z - p_1}U(z) + \dots + \frac{c_n}{z - p_n}U(z).$$

Declaring the states as given and taking inverse z-transform,

$$y(k) = c_1x_1(k) + \dots + c_nx_n(k) = [c_1 \quad \dots \quad c_n]\mathbf{x}(k).$$

From the expression $X_i(z) = \frac{1}{z-p_i}U(z)$, we have $zX_i(z) = p_iX_i(z) + U(z)$. Taking inverse z-transform and forming a system of equations,

$$\begin{aligned}
x_1(k+1) &= p_1x_1(k) + u(k), \\
&\vdots \\
x_n(k+1) &= p_nx_n(k) + u(k),
\end{aligned}$$

we can write it in state-space form as

$$\mathbf{x}(k+1) = \begin{bmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & p_n \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(k).$$

Eventually, we have

$$\mathbf{\Phi}_m = \begin{bmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & p_n \end{bmatrix}, \quad \mathbf{\Gamma}_m = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

$$\mathbf{C}_m = [c_1 \;\; c_2 \;\; \dots \;\; c_n], \quad \mathbf{D}_m = 0.$$

- **Problem 5 [15 Points]**
  Transfer function of a plant is given by

$$H(s) = \frac{10}{(s+1)(s+2)}.$$

(a) Convert $H(s)$ into Observer Canonical form of discrete-time state space representation, using Matched Pole-Zero method (MPZ, recall from the mid-term and from the lecture on Nov. 11)

Poles at $s = -1$ and $s = -2$ in s-plane map on to poles at $z = e^{-0.01} = 0.99$ and $z = e^{-0.02} = 0.98$, for $T = 0.01$ s. Hence, let

$$G(z) = K\frac{(z+1)^2}{(z-0.99)(z-0.98)}.$$

$$\text{DC gain for } H(s) = \lim_{s\to 0} H(s) = 5.$$

$$\text{DC gain for } G(z) = \lim_{z\to 1} G(z)z = 20000K.$$

Equating the DC gains leads to $K = 2.5 \times 10^{-4}$. Therefore,

$$G(z) = 2.5\times 10^{-4}\frac{(z+1)^2}{(z-0.99)(z-0.98)} = 2.5\times 10^{-4}\frac{z^2+2z+1}{z^2-1.97z+0.97}.$$

$$\mathbf{\Phi}_o = \begin{bmatrix} 1.97 & 1 \\ -0.97 & 0 \end{bmatrix}, \mathbf{\Gamma}_o = \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \end{bmatrix} = \begin{bmatrix} 9.925\times 10^{-4} \\ 7.5\times 10^{-6} \end{bmatrix}, \mathbf{C}_o = [1 \;\; 0], \mathbf{D}_o = 2.5{\times}10^{-6}.$$

(b) Find out the polynomial $\alpha(s)$ to yield a 10% overshoot with a settling time of 0.5 s. Discretize $\alpha(s)$ through MPZ method, and design a discrete-time predictive observer for the system.

For $M_p = 10\%$ and $t_s = 0.5$, $\zeta = 0.6$ and $\omega_n = 15.33$. The characteristic polynomial is

$$s_2 + \zeta\omega_n s + \omega_n^2 = s^2 + 18.4s + 235.$$

After mapping its poles to z-domain, with $T = 0.01$, the characteristic equation becomes

$$z^2 - 1.811z + 0.832.$$

Therefore,

$$\mathbf{L}_o = \begin{bmatrix} \alpha_1 - a_1 \\ \alpha_2 - a_2 \end{bmatrix} = \begin{bmatrix} -1.811 + 1.97 \\ 0.832 - 0.97 \end{bmatrix} = \begin{bmatrix} 0.159 \\ -0.138 \end{bmatrix}$$