EE 561: Digital Control Systems

Problem Set 4

Due on Mon. 4th Nov. in class — *Fall 2013*

- **Problem 1 [30 Points]**

A certain system with state $\mathbf{x}$ is described by the state matrices

$$\mathbf{F} = \begin{bmatrix} -2 & 0 \\ -2 & 1 \end{bmatrix}, \; \mathbf{G} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \; \mathbf{H} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; \mathbf{J} = 0,$$

(a) Compute the eigenvalues and eigenvectors of the matrix $\mathbf{F}$. Write down a diagonal matrix $\mathbf{\Lambda}$ and a matrix $\mathbf{V}$ such that $\mathbf{F} = \mathbf{V\Lambda V^{-1}}$. (Hint: recall 'eigendecomposition' from your Linear Algebra course.)

For eigenvalues of $\mathbf{F}$,

$$\det(\mathbf{F} - \lambda \mathbf{I}) = 0,$$
$$\implies \lambda_1 = 1, \; \lambda_2 = -2.$$

Let $\mathbf{v}_1 = [v_{11} \;\; v_{12}]^T$ and $\mathbf{v}_2 = [v_{21} \;\; v_{22}]^T$ be the corresponding eigenvectors.
For $\mathbf{v}_1$,

$$(\mathbf{F} - \lambda_1 \mathbf{I})\mathbf{v}_1 = \mathbf{0},$$
$$\begin{bmatrix} -3 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\implies v_{11} = 0, \text{ and let } v_{12} = k_1.$$

For $\mathbf{v}_2$,

$$(\mathbf{F} - \lambda_2 \mathbf{I})\mathbf{v}_2 = \mathbf{0},$$
$$\begin{bmatrix} 0 & 0 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$-2v_{21} + 3v_{22} = 0.$$

$$\text{Let } v_{22} = k_2, \implies v_{21} = \frac{3}{2}k_2$$

Using $k_1 = 1$ and $k_2 = 2$, we get $\mathbf{v}_1 = [0 \;\; 1]^T$ and $\mathbf{v}_2 = [3 \;\; 2]^T$.
Now,

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix},$$

and,

$$\mathbf{V} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}.$$

(b) Using the results from (a), find out the transformation $\mathbf{T}$ so that if $\mathbf{x} = \mathbf{Tz}$, the state matrices describing the dynamics of $\mathbf{z}$ are in Modal Canonical form.

$$\mathbf{T} = \mathbf{V}.$$

(c) Compute the new matrices $\mathbf{A}, \mathbf{B}, \mathbf{C},$ and $\mathbf{D}$.

$$\mathbf{A} = \mathbf{T}^{-1}\mathbf{FT} = \mathbf{V}^{-1}\mathbf{FV} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix},$$
$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{G} = [-1 \;\; 1]^T,$$
$$\mathbf{C} = \mathbf{HT} = [0 \;\; 3],$$
$$\mathbf{D} = \mathbf{J} = 0.$$

(d) Compare the matrices $\mathbf{A}$ and $\mathbf{\Lambda}$.

Both are same because

$$\mathbf{F} = \mathbf{TAT}^{-1} = \mathbf{V\Lambda V}^{-1}.$$

(e) Plot the step-response of the system using the following commands in Matlab

```
>>sys = ss(A, B, C, D)
>>step(sys)
```

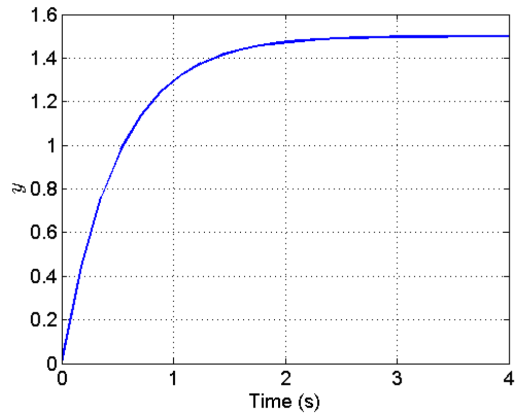

Figure 1: Step response for 1(e).

(f) Write down the total solution $\mathbf{x}(t)$ of the system.

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau. \quad (1)$$

(g) Discretize the total solution with sampling period T to convert the state-equations into the following form

$$\mathbf{x}_{k+1} = \mathbf{\Phi}\mathbf{x}_k + \mathbf{\Gamma}u_k,$$

$$y_k = \mathbf{H}\mathbf{x}_k,$$

where $\mathbf{\Phi}$ and $\mathbf{\Gamma}$ need to be computed.

Let $\mathbf{x}(t_0) = \mathbf{x}(kT)$ and $\mathbf{x}(t) = \mathbf{x}((k+1)T)$. Substituting these into (1), assuming $u(t) = u(kT)$ between kT and $(k+1)T$, and setting $\eta = kT + T - \tau$, we get

$$\mathbf{x}((k+1)T) = e^{\mathbf{A}T}\mathbf{x}(kT) + \int_0^T e^{\mathbf{A}\eta}d\eta\mathbf{B}u(kT),$$

$$\mathbf{x}_{k+1} = e^{\mathbf{A}T}\mathbf{x}_k + \int_0^T e^{\mathbf{A}\eta}d\eta\mathbf{B}u_k.$$

Comparing this result with

$$\mathbf{x}_{k+1} = \mathbf{\Phi}\mathbf{x}_k + \mathbf{\Gamma}u_k,$$

we can deduce that

$$\mathbf{\Phi} = e^{\mathbf{A}T} = \begin{bmatrix} e^T & 0 \\ 0 & e^{-2T} \end{bmatrix},$$

$$\mathbf{\Gamma} = \int_0^T e^{\mathbf{A}\eta}d\eta\mathbf{B} = \begin{bmatrix} e^{\eta}|_0^T & 0 \\ 0 & -0.5e^{-2\eta}|_0^T \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^T - 1 & 0 \\ 0 & -0.5(e^{-2T} - 1) \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

$$\implies \mathbf{\Gamma} = \begin{bmatrix} 1 - e^T \\ 0.5(1 - e^{-2T}) \end{bmatrix}.$$

(h) Find out the discretized transfer function $H(z)$ of the system.

$$H(z) = \mathbf{C}(z\mathbf{I} - \mathbf{\Phi})^{-1}\mathbf{\Gamma} = \frac{1.5(1 - e^{-2T})}{z - e^{-2T}}.$$

(i) Plot the step-response of the discretized system with $T = 0.01$ s. You can use the Matlab function `c2d(sys,T)` to discretize the system.

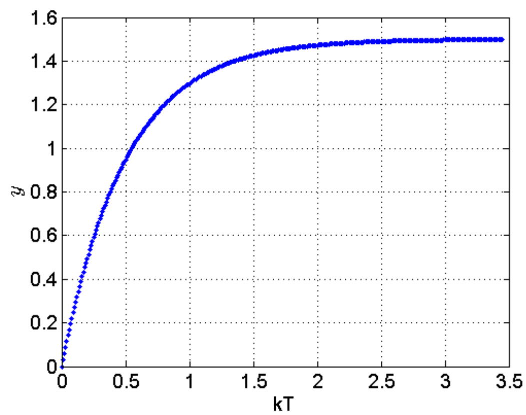

Figure 2: Discrete step response for 1(i).

(j) Compare the step-response obtained in (i) with the one obtained in (e). Comment on any similarities and/or differences.

Step-responses are very similar without any observable differences, because $T << $ time constant. However, discrete-time step response is slightly faster.

- **Problem 2 [20 Points]**

Transfer function of a plant is given by

$$G(s) = \frac{10}{(s+1)(s+2)}.$$

(a) Using 'comparing the coefficients' method, design a state feedback (controller) for this plant to yield a 15% overshoot with a settling time of 0.5 s.

$$\sigma = \frac{4.6}{t_s} = \frac{4.6}{0.5} = 9.2.$$

$$\omega_n = \frac{\sigma}{\zeta} = \frac{9.2}{0.5} = 18.4.$$

We can now write down the characteristic equations as

$$\begin{aligned} & s^2 + 2\zeta\omega_n s + \omega_n^2, \\ = \; & s^2 + 18.4s + 338.56. \end{aligned}$$

First, we write the state-space matrices in Control canonical form as

$$\mathbf{A} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \; \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; \mathbf{C} = \begin{bmatrix} 0 & 10 \end{bmatrix}, \; \mathbf{D} = 0,$$

Then, let $\mathbf{K} = [k_1 \;\; k_2]$. For this $\mathbf{K}$, $\det(s\mathbf{I} - (\mathbf{A} - \mathbf{BK})) = s^2 + s(3 + k_1) + 2 + k_2$. By matching its coefficients with the coefficients of the characteristic equation found above, we calculate $k_1 = 15.4$ and $k_2 = 336.56$.

(b) Repeat (a) using Ackermann's control formula for pole placement.

$$\mathcal{C}^{-1} = [\mathbf{B} \ \ \mathbf{AB}]^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

$$\alpha(\mathbf{A}) = \mathbf{A}^2 + 18.4\mathbf{A} + 338.56\mathbf{I} = \begin{bmatrix} 290.36 & -30.8 \\ 15.4 & 336.56 \end{bmatrix}.$$

In Ackermann's control formula for a second order system,

$$\mathbf{K} = [0 \ \ 1]\mathcal{C}^{-1}\alpha(A) = [15.4 \ \ 336.56].$$

(c) Plot the step-response of the system and verify that the required specifications have been successfully met.

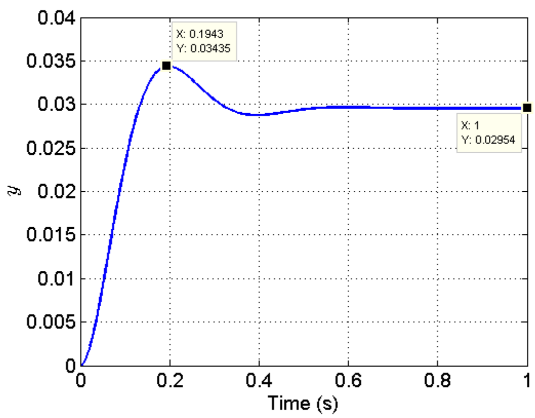

Figure 3: Step response for 2(c).

In the graph, we can observe that the oscillations take roughly 0.5 s to settle. Whereas percentage overshoot $= \frac{0.03435-0.02954}{0.02954} \times 100 = 16.28\% \simeq 15\%$.

- **Problem 3 [10 Points]**

Determine whether the system with state $\mathbf{x}$ and described by the state matrices

$$\mathbf{F} = \begin{bmatrix} -2 & -1 & -3 \\ 0 & -2 & 1 \\ -7 & -8 & -9 \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \ \mathbf{H} = \begin{bmatrix} 4 & 6 & 8 \end{bmatrix}, \ \mathbf{J} = 0,$$

is observable. Verify your answer using the following commands in Matlab

```
>>ObsMat = obsv(F, H)
>>Rank = rank(ObsMat)
```

The observability matrix

$$\mathcal{O} = \begin{bmatrix} \mathbf{H} \\ \mathbf{HF} \\ \mathbf{HF^2} \end{bmatrix} = \begin{bmatrix} 4 & 6 & 8 \\ -64 & -80 & -78 \\ 674 & 848 & 814 \end{bmatrix}.$$

$\det(\mathcal{O}) = -1576 \neq 0$. Hence, the system is observable.

Matlab command gives the rank of the observability matrix as 3. This means that the observability matrix is full rank, hence the system is observable.

- **Problem 4 [10 Points]**
The plant's transfer function of a model for the body's blood glucose level is given below,

$$G(s) = \frac{407(s + 0.916)}{(s + 1.27)(s + 2.69)}.$$

Design an observer for this plant to meet the specifications $\zeta = 0.7$ and $\omega_n = 100$ for the closed-loop system.

The characteristic equation,

$$\begin{aligned} & s^2 + 2\zeta\omega_n s + \omega_n^2 \\ = \; & s^2 + 140s + 10000. \end{aligned}$$

From

$$G(s) = \frac{407(s + 0.916)}{(s + 1.27)(s + 2.69)} = \frac{407s + 372.812}{s^2 + 3.96s + 3.4163},$$

we can write down the state-space matrices for the Control canonical form as

$$\mathbf{A} = \begin{bmatrix} -3.96 & -3.4163 \\ 1 & 0 \end{bmatrix}, \; \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; \mathbf{C} = \begin{bmatrix} 407 & 372.812 \end{bmatrix}, \; \mathbf{D} = 0.$$

Now, let $\mathbf{L} = [l_1 \;\; l_2]^T$, and

$$\mathcal{O}^{-1} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix}^{-1} = \begin{bmatrix} 407 & 3728 \\ -1238.9 & -1390.4 \end{bmatrix}^{-1} = \begin{bmatrix} -0.0003431 & -0.0009199 \\ 0.0003057 & 0.0001004 \end{bmatrix}.$$

$$\alpha(\mathbf{A}) = \mathbf{A}^2 + 140\mathbf{A} + 10000\mathbf{I} = \begin{bmatrix} 9457.9 & -464.8 \\ 136 & 9996.6 \end{bmatrix}.$$

Using Ackermann's observer formula for a second order system,

$$\mathbf{L} = \alpha(A)\mathcal{O}^{-1}[0 \;\; 1]^T = [0.0326 \quad -0.1395]^T.$$

- **Problem 5 [30 Points]**
The plant for the antenna azimuth position control system is given below,

$$G(s) = \frac{1325}{s(s + 1.71)(s + 100)}.$$

A controller needs to be designed to meet certain design specifications. And because the state variables of the plant are not accessible, an observer also needs to be designed to estimate the states.

(a) Write down the state matrices $\mathbf{A}, \mathbf{B}, \mathbf{C},$ and $\mathbf{D}$ for the plant.
From

$$G(s) = \frac{1325}{s(s + 1.71)(s + 100)} = \frac{1325}{s^3 + 101.71s^2 + 171s},$$

we can write down the state-space matrices for the Control canonical form as

$$\mathbf{A} = \begin{bmatrix} -101.71 & -171 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; \mathbf{C} = \begin{bmatrix} 0 & 0 & 1325 \end{bmatrix}, \; \mathbf{D} = 0.$$

(b) Draw a block diagram of the complete control system including plant, controller and observer.

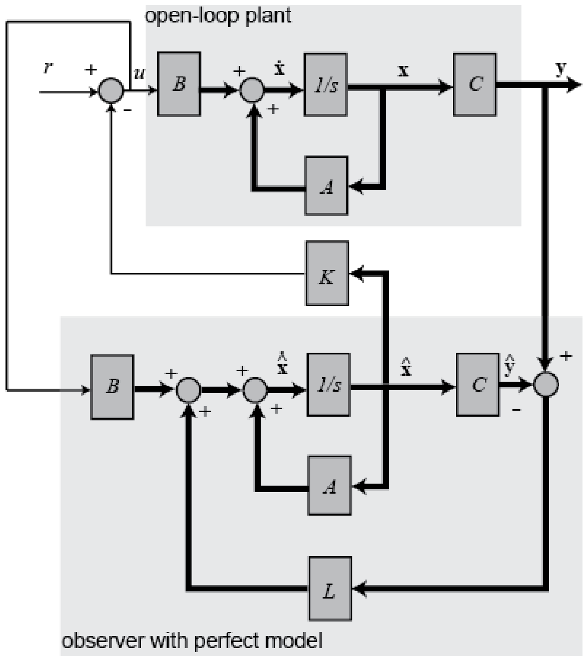

Figure 4: Block diagram for 5(b).

(c) Determine the controllability of the system. Show your working clearly.

$$\mathcal{C} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 1 & -101.71 & 10174 \\ 0 & 1 & -101.71 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\det(\mathcal{C}) = 1\times1\times1 = 1,$ because the matrix is triangular. Hence, the system is controllable.

(d) If the system is controllable, design a controller to yield a 10% overshoot and a settling time of 1 s. Place the third pole 10 times as far from the imaginary axis as the second-order dominant pole.

$$\begin{aligned} \text{Overshoot} &= e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.1, \\ &\implies \zeta \simeq 0.6, \\ \text{and } \omega_n &= \frac{4.6}{t_s\zeta} = 7.67. \end{aligned}$$

The characteristic equation with two poles is

$$\begin{aligned} & s^2 + 2\zeta\omega_n s + \omega_n^2, \\ = \; & s^2 + 9.2s + 58.8, \end{aligned}$$

which has poles at $-4.6 \pm 6.14j$. By placing the third pole at $-4.6 \times 10 = -46$, the final characteristic equation becomes

$$\begin{aligned} & (s+46)(s^2 + 9.2s + 58.8), \\ = \; & s^3 + 55.2s^2 + 482s + 2704.8. \end{aligned}$$

Now, let $\mathbf{K} = [k_1 \quad k_2 \quad k_3]$.

$$\begin{aligned} k_1 &= \alpha_1 - a_1 = 55.2 - 101.71 = -46.51, \\ k_2 &= \alpha_2 - a_2 = 482 - 171 = 311, \\ k_3 &= \alpha_3 - a_3 = 2704.8 - 0 = 2704.8. \end{aligned}$$

(e) Determine the observability of the system. Show your working clearly.

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA^2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1325 \\ 0 & 1325 & 0 \\ 1325 & 0 & 0 \end{bmatrix}.$$

$\det(\mathcal{O}) = -1325^3 \neq 0$. Hence, the system is observable.

(f) If the system is observable, design an observer whose transient response has 10% overshoot and a natural frequency 10 times as great as the natural frequency of the response in (d). Place the third pole 10 times as far from the imaginary axis as the observer's second-order dominant pole.

For 10% overshoot, $\zeta \simeq 0.6$.
$\omega_n = 10 \times 7.67 = 76.7$.
The characteristic equation with two poles is

$$\begin{aligned} & s^2 + 2\zeta\omega_n s + \omega_n^2, \\ = \; & s^2 + 92s + 5883, \end{aligned}$$

which has poles at $-46 \pm 61.4j$. By placing the third pole at $-46 \times 10 = -460$, the final characteristic equation becomes

$$\begin{aligned} & (s + 460)(s^2 + 92s + 5883), \\ = \; & s^3 + 552s^2 + 48203s + 2706180. \end{aligned}$$

Let $\mathbf{L} = [l_1 \;\; l_2 \;\; l_3]^T$.

$$\mathcal{O}^{-1} = \begin{bmatrix} 0 & 0 & 1325 \\ 0 & 1325 & 0 \\ 1325 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0.0007547 \\ 0 & 0.0007547 & 0 \\ 0.0007547 & 0 & 0 \end{bmatrix}.$$

$$\alpha(\mathbf{A}) = \mathbf{A}^3 + 552\mathbf{A}^2 + 48203\mathbf{A} + 2706180\mathbf{I} = \begin{bmatrix} 2402100 & -381800 & 0 \\ 2200 & 2629200 & 0 \\ 500 & 48000 & 2706200 \end{bmatrix}.$$

Using Ackermann's observer formula for a third order system,

$$\mathbf{L} = \alpha(A)\mathcal{O}^{-1}[0 \;\; 0 \;\; 1]^T = [1813 \;\; 1.7 \;\; 0.3]^T.$$

(g) Using Matlab, plot the step-response of the system using zero initial conditions. Also plot the estimated output $\hat{y}$ on the same graph.

With $u = r - \mathbf{K}\hat{\mathbf{x}}$, the plant's state equations are

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{BK}\hat{\mathbf{x}} + \mathbf{B}r, \qquad (2)$$
$$y = \mathbf{C}\mathbf{x} - \mathbf{DK}\hat{\mathbf{x}} + \mathbf{D}r, \qquad (3)$$

and the observer's state equations are

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{BK})\hat{\mathbf{x}} + \mathbf{B}r + \mathbf{L}(y - \hat{y}), \qquad (4)$$
$$\hat{y} = (\mathbf{C} - \mathbf{DK})\hat{\mathbf{x}} + \mathbf{D}r. \qquad (5)$$

With zero initial conditions, $\mathbf{x} = \hat{\mathbf{x}}$, which implies $y = \hat{y}$ for all $t > 0$. Now, $\mathbf{A}_{\text{new}} = \mathbf{A} - \mathbf{BK}$, $\mathbf{B}_{\text{new}} = \mathbf{B}$, $\mathbf{C}_{\text{new}} = \mathbf{C} - \mathbf{DK}$, and $\mathbf{D}_{\text{new}} = \mathbf{D}$. Using these state matrices, the step response plotted is shown in Fig. 5.

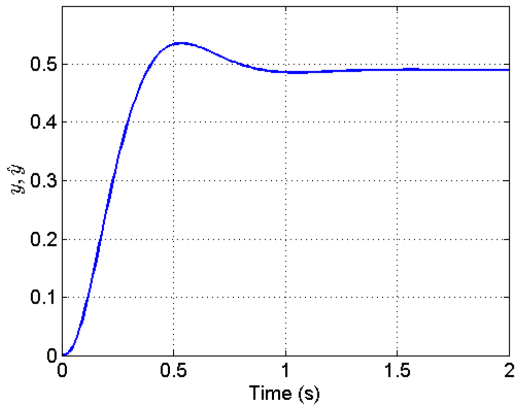

Figure 5: Step response for 5(g).

(h) Now plot the step-response of the system assuming the initial condition $x_1 = 0.006$ at $t = 0$. Also plot the estimated output $\hat{y}$ on the same graph using zero initial conditions.
Let $\mathbf{x} - \hat{\mathbf{x}} = \mathbf{e_x}$, and $y - \hat{y} = e_y$.
Eq. (3) − eq. (5),

$$y - \hat{y} = \mathbf{C}(\mathbf{x} - \hat{\mathbf{x}}),$$
$$\implies e_y = \mathbf{C}\mathbf{e_x}.$$

Eq. (2) − eq. (4),

$$\dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{A}(\mathbf{x} - \hat{\mathbf{x}}) - \mathbf{L}(y - \hat{y}) = (\mathbf{A} - \mathbf{LC})(\mathbf{x} - \hat{\mathbf{x}}),$$

$$\dot{\mathbf{e}}_\mathbf{x} = (\mathbf{A} - \mathbf{LC})\mathbf{e_x}.$$

For capturing the error dynamics, let's form a new system with

$$\mathbf{A_e} = \mathbf{A} - \mathbf{LC},\ \mathbf{B_e} = \mathbf{0},\ \mathbf{C_e} = \mathbf{C},\ \text{and } \mathbf{D} = 0.$$

Solving this new system for e_y with initial condition $\mathbf{e_x}(0) = [0.006 \quad 0 \quad 0]^T$, and adding it to $\hat{y}$ (the solution in (g)), we can recover y because $y = \hat{y} + e_y$.

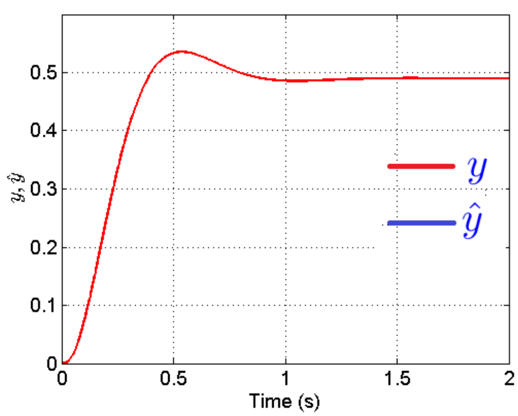

Figure 6: Step response for 5(h).

Magnified plot near origin is shown below.

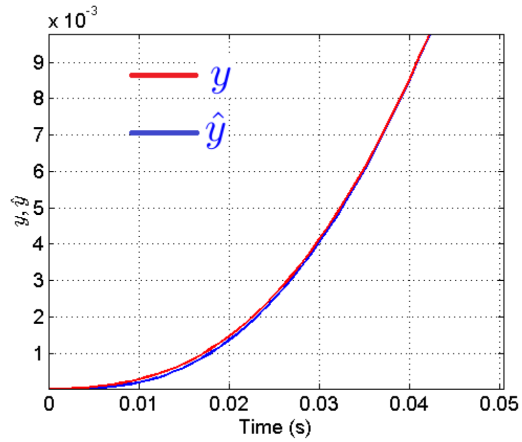

Figure 7: Magnified plot for 5(h).

(i) How long does it take for the output error to decay below 1%?

The error remains less than 1% for all $t > 0$.