EE 561: Digital Control Systems

Problem Set 2: Solution

Due on Wed. 25th Sept. *Fall 2013*

- **Problem 1**
Check the following for (internal) stability. [Hint: Analyze the characteristic equation.]

  (a) $u_k = 0.5u_{k-1} - 0.3u_{k-2}$
  Taking the z-transform, we get
  $$U(z)\left(1 - 0.5z^{-1} + 0.3z^{-2}\right) = 0,$$
  $$U(z)\left(z^2 - 0.5z + 0.3\right) = 0.$$
  The characteristic equation turns out to be
  $$z^2 - 0.5z + 0.3 = 0,$$
  with roots $z_{1,2} = 0.25 \pm 0.49j$. $|z_{1,2}| = \sqrt{0.25^2 + 0.49^2} \simeq 0.55$, which is less than 1. Both roots lie inside the unit circle, hence the system is stable.

  (b) $u_k = 1.6u_{k-1} - u_{k-2}$
  The characteristic equation is
  $$z^2 - 1.6z + 1 = 0,$$
  with roots $z_{1,2} = 0.8 \pm 0.6j$. $|z_{1,2}| = 1$. Roots lie on the unit circle, so the system is marginally (neutrally) stable.

  (c) $u_k = 0.8u_{k-1} + 0.4u_{k-2}$
  The characteristic equation is
  $$z^2 - 0.8z - 0.4 = 0,$$
  with roots $z_1 = 1.148$, $z_2 = -0.348$. $|z_1| > 1$. One root lie outside the unit circle, so the system is unstable.

- **Problem 2**
The first-order system $\dfrac{z-\alpha}{(1-\alpha)z}$ has a zero at $z = \alpha$.

  (a) Plot the step response for this system for $\alpha = 0.8, 0.9, 1.1, 1.2, 2$.
  Step responses are shown in Fig. 1.

  (b) Plot the overshoot of the system on the same coordinates as those appearing in Fig. 3 for $-1 < \alpha < 1$.
  Overshoot is shown in Fig. 2.

  (c) In what way is the step response of this system unusual for $\alpha > 1$?
  For $\alpha > 1$, responses undershoot instead of overshooting, before settling to the stable point.

- **Problem 3**
Show that the characteristic equation
$$z^2 - 2r\cos(\theta)z + r^2$$
has the roots
$$z_{1,2} = re^{\pm j\theta}.$$
Applying the quadratic roots formula, we get
$$z_{1,2} = r\cos\theta \pm r\sqrt{\cos^2\theta - 1} = r\cos\theta \pm r\sqrt{-\sin^2\theta} = r(\cos\theta \pm j\sin\theta) = re^{\pm j\theta}$$

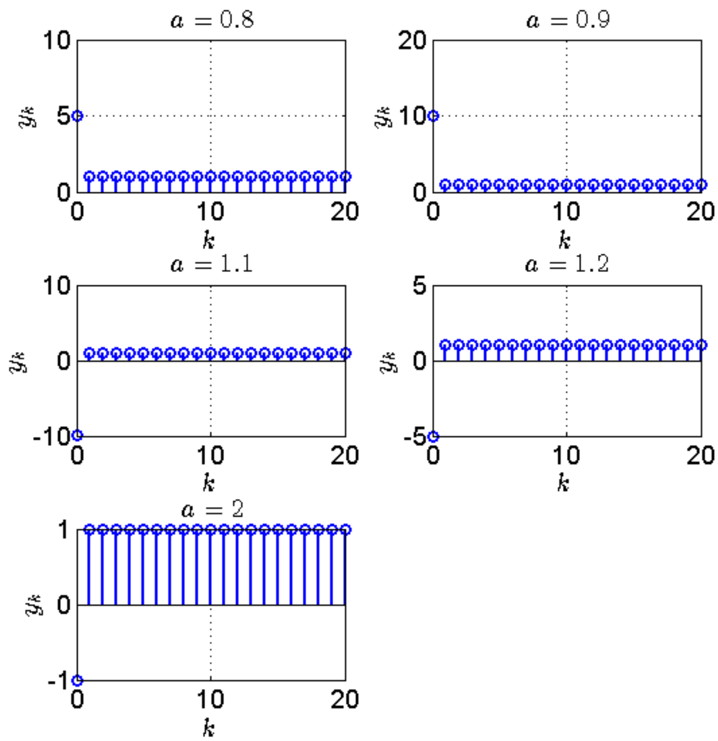

Figure 1: Step responses with changing position of zero z_2.

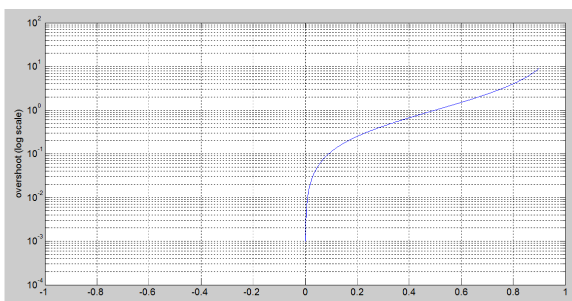

Figure 2: Variation of percentage overshoot with changing position of zero z_2.

- **Problem 4**
  For a second-order system with damping ratio 0.5 and poles at an angle in the z-plane of $\theta = 30°$, what percent overshoot to a step would you expect if the system had a zero at $z_2 = 0.6$? [Hint: You can either analyze it geometrically, or you can also estimate it from the graph in Fig. 3.]

  In the graph shown in Fig. 3, curves for $\theta = 18°$ and $\theta = 45°$ are shown. We can draw an approximate curve (drawn in blue) for $\theta = 30°$ between those curves. From the new curve, we can estimate the percentage overshoot at $z_2 = 0.6$ as roughly 40%.

- **Problem 5**
  Sketch the inverse z-transform, f_k, for each of the following transforms, for up to at least three time constants. You can also use an inverse z-transform table.

  (a) $F(z) = \dfrac{1}{1+z^{-2}}$, $\quad |z| > 1;$

  Inverse transform f_k can be considered as the impulse response of a transfer function $H(z)$, considering $H(z) = \dfrac{Y(z)}{U(z)} = F(z)$.

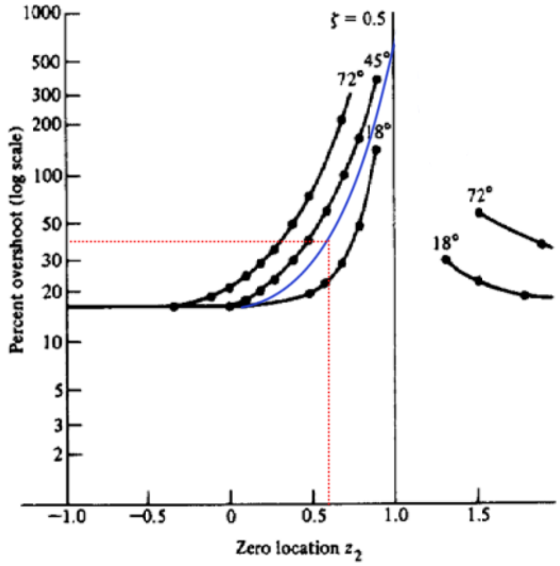

Figure 3:

So, if

$$\frac{Y(z)}{U(z)} = \frac{1}{1+z^{-2}},$$
$$\Rightarrow y_k = u_k - y_{k-2}.$$

Now considering $u_k = \delta[k]$, y_k equals f_k,

$$\Rightarrow f_k = \delta[k] - f_{k-2}.$$

For a causal system $f_k = 0$ for $k < 0$, so

$$\begin{aligned}
f_0 &= \delta[0] - f_{-2} = 1 - 0 = 1,\\
f_1 &= \delta[1] - f_{-1} = 0 - 0 = 0,\\
f_2 &= \delta[2] - f_0 = 0 - 1 = -1,\\
f_3 &= \delta[3] - f_1 = 0 - 0 = 0,\\
f_4 &= \delta[4] - f_2 = 0 - (-1) = 1,\\
f_5 &= \delta[5] - f_3 = 0 - 0 = 0,\\
&.\\
&.\\
&.
\end{aligned}$$

(b) $F(z) = \dfrac{z(z-1)}{z^2 - 1.25z + 0.25}$, $\quad |z| > 1$;

Considering

$$\frac{Y(z)}{U(z)} = \frac{z(z-1)}{z^2 - 1.25z + 0.25},$$
$$\Rightarrow y_k = u_k - u_{k-1} + 1.25y_{k-1} - 0.25y_{k-2}.$$

Now considering $u_k = \delta[k]$, y_k equals f_k,

$$\Rightarrow f_k = \delta[k] - \delta[k-1] + 1.25f_{k-1} - 0.25f_{k-2}.$$

For a causal system $f_k = 0$ for $k < 0$, so

$$
\begin{aligned}
&f_0 = \delta[0] - \delta[-1] + 1.25f_{-1} - 0.25f_{-2} = 1 - 0 = 1,\\
&f_1 = \delta[1] - \delta[0] + 1.25f_0 - 0.25f_{-1} = 0 - 1 + 1.25 - 0 = 0.25,\\
&f_2 = \delta[2] - \delta[1] + 1.25f_1 - 0.25f_0 = 0 - 0 + 1.25(0.25) - 0.25(1) = 0.0625,\\
&f_3 = \delta[3] - \delta[2] + 1.25f_2 - 0.25f_1 = 0 - 0 + 1.25(0.0625) - 0.25(0.25) = 0.0156,\\
&f_4 = \delta[4] - \delta[3] + 1.25f_3 - 0.25f_2 = 0 - 0 + 1.25(0.0156) - 0.25(0.0156) = 0.0039,\\
&f_5 = \delta[5] - \delta[4] + 1.25f_4 - 0.25f_3 = 0 - 0 + 1.25(0.0039) - 0.25(0.0156) = 0.00097656,\\
&.\\
&.\\
&.
\end{aligned}
$$

(c) $F(z) = \dfrac{z}{z^2 - 2z + 1}, \qquad |z| > 1;$

Considering

$$\frac{Y(z)}{U(z)} = \frac{z}{z^2 - 2z + 1},$$

$$\Rightarrow y_k = u_{k-1} + 2y_{k-1} - y_{k-2}.$$

Now considering $u_k = \delta[k]$, y_k equals f_k,

$$\Rightarrow f_k = \delta[k-1] + 2f_{k-1} - f_{k-2}.$$

For a causal system $f_k = 0$ for $k < 0$, so

$$
\begin{aligned}
&f_0 = \delta[-1] + 2f_{-1} - f_{-2} = 0,\\
&f_1 = \delta[0] + 2f_0 - f_{-1} = 1 + 0 = 1,\\
&f_2 = \delta[1] + 2f_1 - f_0 = 0 + 2(1) - 0 = 2,\\
&f_3 = \delta[2] + 2f_2 - f_1 = 0 + 2(2) - 1 = 3,\\
&f_4 = \delta[3] + 2f_3 - f_2 = 0 + 2(3) - 2 = 4,\\
&f_5 = \delta[4] + 2f_4 - f_3 = 0 + 2(4) - 3 = 5,\\
&.\\
&.\\
&.
\end{aligned}
$$

(d) $F(z) = \dfrac{z}{(z - \frac{1}{2})(z - 2)}, \qquad |z| > 2.$

Considering

$$\frac{Y(z)}{U(z)} = \frac{z}{(z - \frac{1}{2})(z - 2)},$$

$$\Rightarrow y_k = u_{k-1} + \frac{5}{2}y_{k-1} - y_{k-2}.$$

Now considering $u_k = \delta[k]$, y_k equals f_k,

$$\Rightarrow f_k = \delta[k-1] + \frac{5}{2}f_{k-1} - f_{k-2}.$$

For a causal system $f_k = 0$ for $k < 0$, so

$$
\begin{aligned}
f_0 &= \delta[-1] + \frac{5}{2} f_{-1} - f_{-2} = 0, \\
f_1 &= \delta[0] + \frac{5}{2} f_0 - f_{-1} = 1 + 0 = 1, \\
f_2 &= \delta[1] + \frac{5}{2} f_1 - f_0 = 0 + 5/2(1) - 0 = 5/2, \\
f_3 &= \delta[2] + \frac{5}{2} f_2 - f_1 = 0 + 5/2(5/2) - 1 = 21/4, \\
f_4 &= \delta[3] + \frac{5}{2} f_3 - f_2 = 0 + 5/2(21/4) - 5/2 = 85/8, \\
f_5 &= \delta[4] + \frac{5}{2} f_4 - f_3 = 0 + 5/2(85/8) - 21/4 = 341/16, \\
&. \\
&. \\
&.
\end{aligned}
$$

Plots of f_k for parts (a)-(d) are shown in Fig. 4.

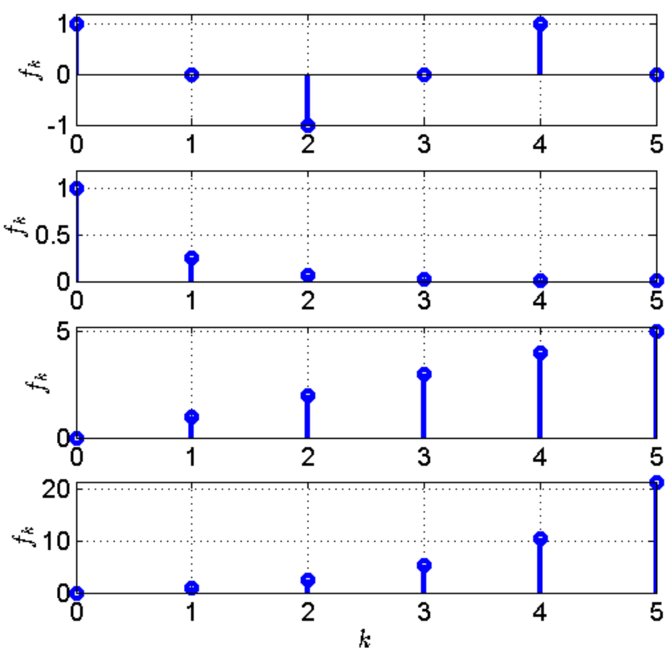

Figure 4: Graphs for Problem 5.

- **Problem 6**
  Use the z-transform to solve the difference equation

$$y_k - 3y_{k-1} + 2y_{k-2} = 2u_{k-1} - 2u_{k-2}.$$

$$u_k = \begin{cases} k, & k \geq 0, \\ 0, & k < 0, \end{cases}$$

$$y_k = 0, \quad k < 0.$$

By taking $z - transform$ of the difference equation with zero initial conditions, we determine the transfer function as

$$\frac{Y(z)}{U(z)} = \frac{z(z^{-1} - z^{-2})}{1 - 3z^{-1} + 2z^{-2}} = \frac{2(z-1)}{z^2 - 3z + 2} = \frac{2(z-1)}{(z-1)(z-2)} = \frac{2}{z-2}.$$

For the given u_k, we look up its z-transform from the table as $U(z) = \frac{z}{(z-1)^2}$. This leads to

$$Y(z) = \left(\frac{Y(z)}{U(z)}\right) U(z) = \frac{2z}{(z-2)(z-1)^2}.$$

Splitting it up into partial fractions,

$$Y(z) = \frac{2z}{z-2} - \frac{2z}{z-1} - \frac{2z}{(z-1)^2},$$

and taking the inverse z-transform, we finally get

$$y_k = 2(2^k - 1 - k), \quad \text{for } k \geq 0.$$

- **Problem 7**
  For this problem, refer to section 6.6.1 of Signals and Systems (2e) by Oppenheim et. al..
  For the first-order causal LTI system described by the difference equation (warning: this is not the same equation that appears in section 6.6.1)

$$y_k + ay_{k-1} = ax_k.$$

  (a) Find out the z-transform $H(z)$ of the system.
  The z-transform

$$\frac{Y(z)}{X(z)} = H(z) = \frac{a}{1 + az^{-1}}, \tag{1}$$

  (b) Using a suitable substitution to the z-transform found in (a), find out the frequency response $H(e^{j\omega})$ of the system.
  Substituting $z = e^{j\omega}$ into (1),

$$H(e^{j\omega}) = \frac{a}{1 + ae^{-j\omega}}.$$

  (c) For $a = -3/2, -1/3, 0, 1/3,$ and $3/2$,

    (i) Plot the step responses of the system.
    See Fig. 5.

    (ii) Plot the magnitude responses of the system. Comment on the variation in shape of the graph with the variation in a.
    See Fig. 6.

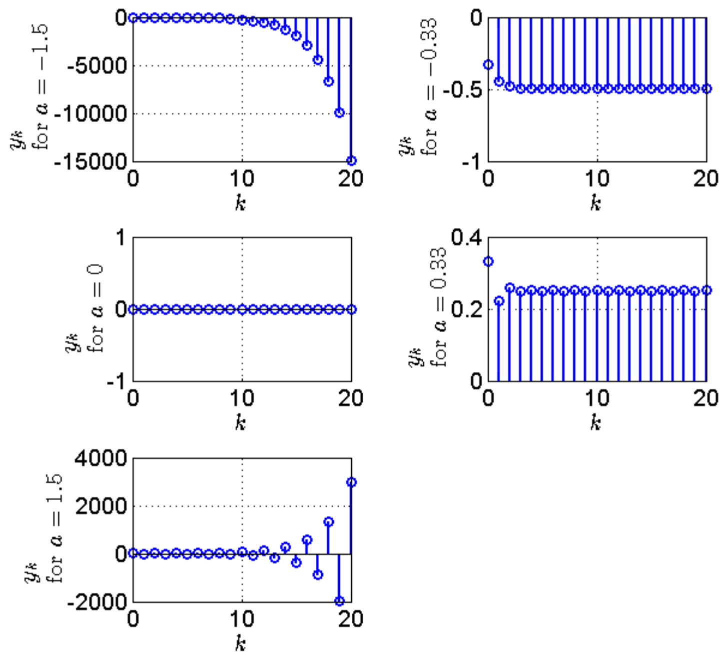

Figure 5: Step responses for Problem 7(c).

- **Problem 8**

  For this problem, refer to section 6.6.2 of Signals and Systems (2e) by Oppenheim et. al.. For the second-order causal LTI system described by

$$y_k - 2r\cos(\theta)y_{k-1} + r^2 y_{k-2} = x_k,$$

  with $0 \leq \theta \leq \pi$.

  (a) Find out the z-transform $H(z)$ of the system.

  The z-transform

$$H(z) = \frac{1}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}\,. \tag{2}$$

  (b) Find out the frequency response $H(e^{j\omega})$ of the system.

  Substituting $z = e^{j\omega}$ into (2),

$$H(e^{j\omega}) = \frac{1}{1 - 2r\cos\theta e^{-j\omega} + r^2 e^{-j2\omega}}\,.$$

  (c) For $(r, \theta) = (2/3, 0), (3/2, \pi/4), (1/3, \pi/2), (2/3, 3\pi/4), (2/3, \pi)$,

  (i) Plot the step responses of the system. Also classify each response as either undamped, underdamped, critically damped or overdamped.

  See Fig. 7.

  (ii) Plot the magnitude responses of the system. Comment on the variation in shape of the graph with the variation in r and θ.

  See Fig. 8.

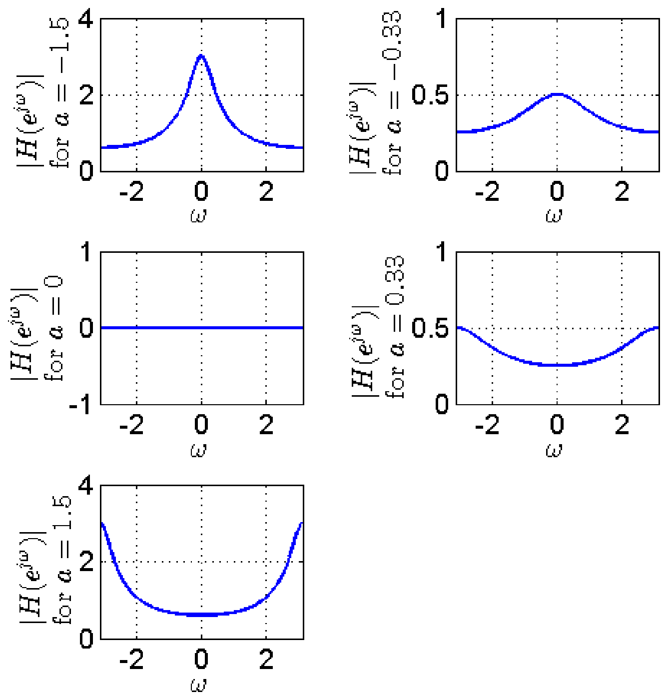

Figure 6: Magnitude responses for Problem 7(c). For $a < 0$, the system acts as a low-pass filter. Whereas, for $a > 0$, the system acts as a high-pass filter. The gain of the filter increases for larger magnitudes of a.

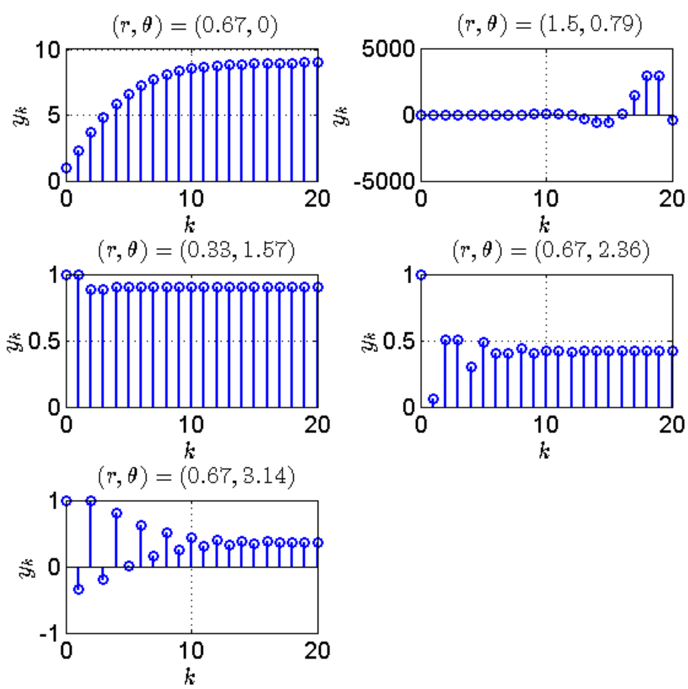

Figure 7: Step responses for Problem 8(c) are critically damped, undamped, overdamped, underdamped, and underdamped respectively.

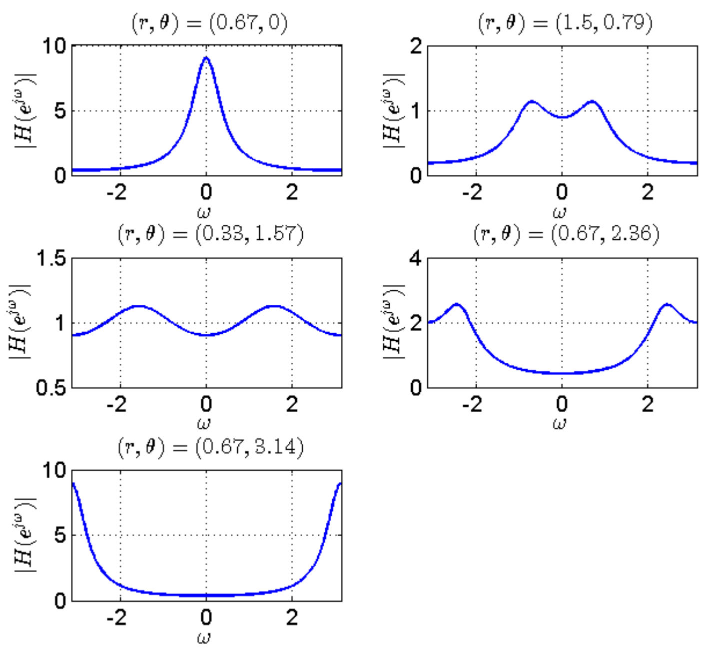

Figure 8: Magnitude responses for Problem 8(c). As θ varies from 0 to π, the system's function shifts from low-pass filtering to high-pass filtering, with band-pass filtering about $\theta = \pi/2$.