EE 561

Assignment 1: Solution

Digital Control Systems *Fall 2013*

- **Problem 1:**

1. Setup a general differential equation for PID control using Euler Forward, Euler Backward, and Trapezoidal approximation rules.

For a PID controller,

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau + k_d e'(t). \tag{1}$$

Method 1
Let $\int_0^t e(\tau) d\tau = \varepsilon(t)$, and $e'(t) = \epsilon(t)$, which leads to

$$u(t) = k_p e(t) + k_i \varepsilon(t) + k_d \epsilon(t). \tag{2}$$

From Euler Forward approximation,

$$\epsilon_k \simeq \frac{e_{k+1} - e_k}{T},$$

$$\varepsilon_k \simeq \varepsilon_{k-1} + T e_{k-1}.$$

From Euler Backward approximation,

$$\epsilon_k \simeq \frac{e_k - e_{k-1}}{T},$$

$$\varepsilon_k \simeq \varepsilon_{k-1} + T e_k.$$

And from Tustin's (Trapezoidal) approximation,

$$\epsilon_k \simeq \frac{2}{T}(e_k - e_{k-1}) - \epsilon_{k-1},$$

$$\varepsilon_k \simeq \varepsilon_{k-1} + \frac{T}{2}(e_k + e_{k-1}).$$

For a discrete PID controller,

$$u_k = k_p e_k + k_i \varepsilon_k + k_d \epsilon_k, \tag{3}$$

and

$$u_{k-1} = k_p e_{k-1} + k_i \varepsilon_{k-1} + k_d \epsilon_{k-1}. \tag{4}$$

Computing (3) $-$ (4) leads to

$$u_k - u_{k-1} = k_p(e_k - e_{k-1}) + k_i(\varepsilon_k - \varepsilon_{k-1}) + k_d(\epsilon_k - \epsilon_{k-1}). \tag{5}$$

Applying **Euler Forward** to (5) yields

$$u_k - u_{k-1} = k_p(e_k - e_{k-1}) + k_i T e_{k-1} + k_d \frac{1}{T}(e_{k+1} - 2e_k + e_{k-1}),$$

$$\Rightarrow u_k - u_{k-1} = k_d \frac{1}{T} e_{k+1} + (k_p - k_d \frac{2}{T}) e_k - (k_p - k_i T - k_d \frac{1}{T}) e_{k-1}. \tag{6}$$

Applying **Euler Backward** to (5) yields

$$u_k - u_{k-1} = k_p(e_k - e_{k-1}) + k_i T e_k + k_d \frac{1}{T}(e_k - 2e_{k-1} + e_{k-2}),$$

$$\Rightarrow u_k - u_{k-1} = (k_p + k_i T + k_d \frac{1}{T})e_k - (k_p + k_d \frac{2}{T})e_{k-1} + k_d \frac{1}{T} e_{k-2}. \tag{7}$$

Applying **Tustin's** (trapezoidal) approximation to (5) yields

$$u_k - u_{k-1} = k_p(e_k - e_{k-1}) + k_i \frac{T}{2}(e_k + e_{k-1}) + k_d \frac{2}{T}(e_k - e_{k-1}) - 2k_d \epsilon_{k-1}, \tag{8}$$

and similarly

$$u_{k-1} - u_{k-2} = k_p(e_{k-1} - e_{k-2}) + k_i \frac{T}{2}(e_{k-1} + e_{k-2}) + k_d \frac{2}{T}(e_{k-1} - e_{k-2}) - 2k_d \epsilon_{k-2}. \tag{9}$$

Now, computing (8) + (9) leads to

$$u_k - u_{k-2} = (k_p + k_i \frac{T}{2} + k_d \frac{2}{T})e_k + (k_i T - k_d \frac{4}{T})e_{k-1} - (k_p - k_i \frac{T}{2} - k_d \frac{2}{T})e_{k-2}. \tag{10}$$

Method 2

Laplace transform $U(s)$ of the output $u(t)$ of a PID controller is given by

$$U(s) = k_p E(s) + k_i \frac{E(s)}{s} + k_d s E(s). \tag{11}$$

Using s-to-z transform for **Euler Forward**,

$$s \to \frac{z-1}{T},$$

$$U(z) = k_p E(z) + k_i \frac{T}{z-1} E(z) + k_d \frac{z-1}{T} E(z),$$

$$(z-1)U(z) = k_p (z-1)E(z) + k_i T E(z) + k_d \frac{(z-1)^2}{T} E(z),$$

$$\Rightarrow zU(z) - U(z) = k_d \frac{1}{T} z^2 E(z) + (k_p - k_d \frac{2}{T}) z E(z) - (k_p - k_i T - k_d \frac{1}{T}) E(z).$$

Taking inverse z-transform,

$$u_{k+1} - u_k = k_d \frac{1}{T} e_{k+2} + (k_p - k_d \frac{2}{T}) e_{k+1} - (k_p - k_i T - k_d \frac{1}{T}) e_k,$$

and decrementing the discrete-time index by 1 finally yields,

$$u_k - u_{k-1} = k_d \frac{1}{T} e_{k+1} + (k_p - k_d \frac{2}{T}) e_k - (k_p - k_i T - k_d \frac{1}{T}) e_{k-1}. \tag{12}$$

Using s-to-z tranform for **Euler Backward**,

$$s \to \frac{z-1}{Tz},$$

$$U(z) = k_p E(z) + k_i \frac{Tz}{z-1} E(z) + k_d \frac{z-1}{Tz} E(z),$$

$$z(z-1)U(z) = k_p z(z-1)E(z) + k_i T z^2 E(z) + k_d \frac{(z-1)^2}{T} E(z),$$

$$\Rightarrow z^2 U(z) - zU(z) = (k_p + k_i T + k_d \frac{1}{T}) z^2 E(z) - (k_p + k_d \frac{2}{T}) z E(z) + k_d \frac{1}{T} E(z).$$

Taking inverse z-transform,

$$u_{k+2} - u_{k+1} = (k_p + k_i T + k_d \frac{1}{T}) e_{k+2} - (k_p + k_d \frac{2}{T}) e_{k+1} + k_d \frac{1}{T} e_k,$$

and decrementing the discrete-time index by 2 finally yields,

$$u_k - u_{k-1} = (k_p + k_i T + k_d \frac{1}{T}) e_k - (k_p + k_d \frac{2}{T}) e_{k-1} + k_d \frac{1}{T} e_{k-2}. \tag{13}$$

Using s-to-z transform for **Tustin's** (trapezoidal) approximation,

$$s \to \frac{2(z-1)}{T(z+1)},$$

$$U(z) = k_p E(z) + k_i \frac{T(z+1)}{2(z-1)} E(z) + k_d \frac{2(z-1)}{T(z+1)} E(z),$$

$$(z+1)(z-1)U(z) = k_p(z+1)(z-1)E(z) + k_i \frac{T}{2}(z+1)^2 E(z) + k_d \frac{2}{T}(z-1)^2 E(z),$$

$$\Rightarrow z^2 U(z) - U(z) = (k_p + k_i \frac{T}{2} + k_d \frac{2}{T}) z^2 E(z) + (k_i T - k_d \frac{4}{T}) z E(z) - (k_p - k_i \frac{T}{2} - k_d \frac{2}{T}) E(z).$$

Taking inverse z-transform,

$$u_{k+2} - u_k = (k_p + k_i \frac{T}{2} + k_d \frac{2}{T}) e_{k+2} + (k_i T - k_d \frac{4}{T}) e_{k+1} - (k_p - k_i \frac{T}{2} - k_d \frac{2}{T}) e_k.$$

and decrementing the discrete-time index by 2 finally yields,

$$u_k - u_{k-2} = (k_p + k_i \frac{T}{2} + k_d \frac{2}{T}) e_k + (k_i T - k_d \frac{4}{T}) e_{k-1} - (k_p - k_i \frac{T}{2} - k_d \frac{2}{T}) e_{k-2}. \quad (14)$$

2. Give pseudo-code to implement each controller.

Euler Forward

Difference equation derived from Euler Forward Method is non-realizable, so its functional pseudo-code cannot be written.

Euler Backward

```
% initialize parameters and states
T = 1;  kp = 100;  ki = 1;  kd = 10;  u = 0;  e_{-1} = 0;  e_{-2} = 0;
% pre-computation of coefficients to save the computation time
k1 = kp + ki * T + kd/T;  k2 = kp + kd * 2/T;  k3 = kd/T;
while(1)
    % read current input values of 'y' and 'r' at A/D
    y = ;  r = ;
    e = y - r;
    u = u + k1 * e - k2 * e_{-1} + k3 * e_{-2};
    e_{-2} = e_{-1};
    e_{-1} = e;
    % wait until the next time interval
    sleep(T);
end while
```

Tustin's Method

```
% initialize parameters and states
T = 1;  kp = 100;  ki = 1;  kd = 10;  u_{-2} = 0;  u_{-1} = 0;  e_{-1} = 0;  e_{-2} = 0;
% pre-computation of coefficients to save the computation time
k1 = kp + ki * T/2 + kd * 2/T;  k2 = ki * T - kd * 4/T;  k3 = kp - ki * T/2 - kd * 2/T;
while(1)
    % read current input values of 'y' and 'r' at A/D
    y = ;  r = ;
    e = y - r;
    u = u_{-2} + k1 * e + k2 * e_{-1} - k3 * e_{-2}
    e_{-2} = e_{-1};
    e_{-1} = e;
    u_{-2} = u_{-1};
    u_{-1} = u;
    % wait until the next time interval
    sleep(T);
end while
```

- **Problem 2:**
  Reproduce the results of Example 4.8 (Franklin) of a PI controller for motor speed control using SIMULINK. Compare analog control with the suggested digital control.

  For T = 0.07,

$$D(z) = 1.4\frac{1.21z - 0.79}{z - 1}.$$

  For T = 0.035,

$$D(z) = 1.4\frac{1.105z - 0.895}{z - 1}.$$

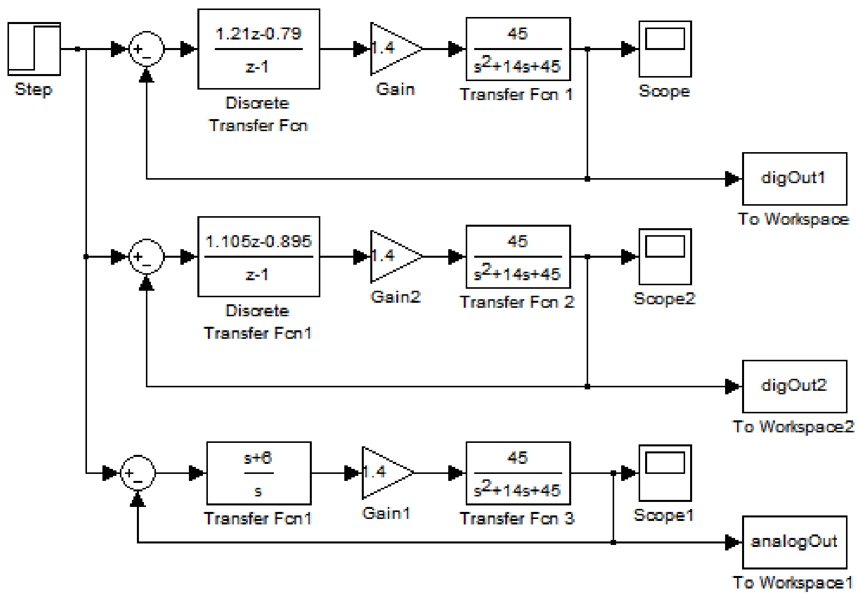

Figure 1: Simulink block diagram.

The step responses from these controllers are shown in Fig. 2, along the step response of the analog controller.

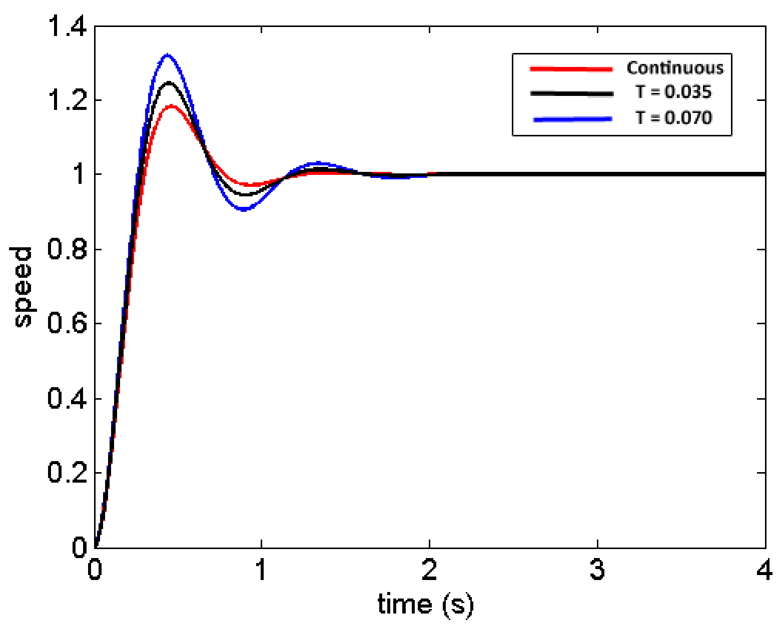

Figure 2: Step responses of an analog and two discretized controllers with different sampling rates.

- **Problem 3:**
Setup and simulate digital controllers to implement the lead compensator $D(s) = 70\frac{s+2}{s+10}$, for the plant $G(s) = \frac{1}{s(s+1)}$, at clock frequencies of 20 Hz and 40 Hz. Compare your results with analog control.

For T=1/20, applying Tustin's method for continuous to discrete transformation, we get

$$D(z) = \frac{58.8z - 53.2}{z - 0.6},$$

and for T=1/40, we get

$$D(z) = \frac{63.78z - 60.67}{z - 0.78}.$$

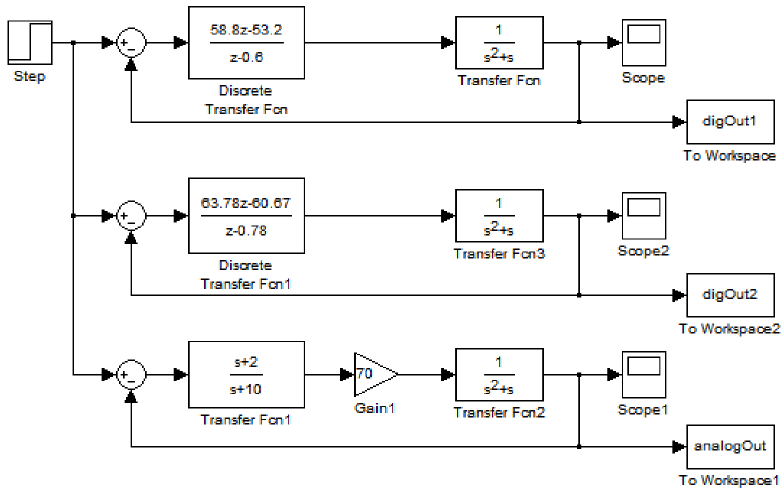

Figure 3: Simulink block diagram.

The step responses from these controllers are shown in Fig. 4, along the step response of the analog controller. We can observe that as the sample time decreases, the response from discrete controllers matches more closely to the analog controller's response.

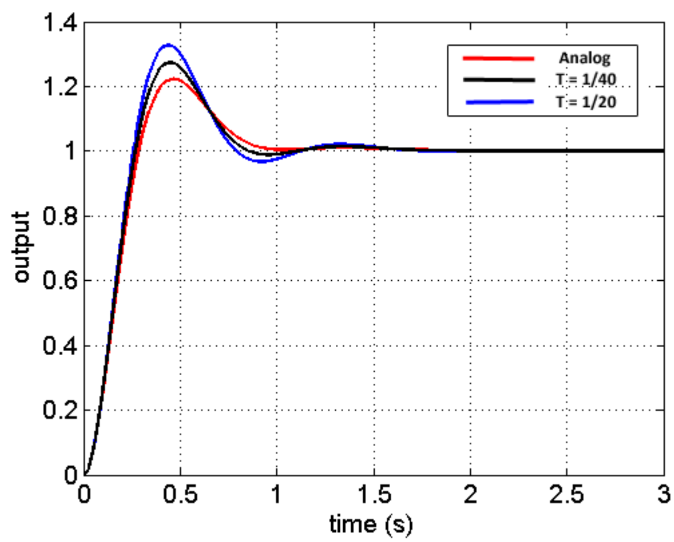

Figure 4: Step responses of an analog and two discretized controllers with different sampling rates.

- **Problem 4:**
Refer to the previous problem ($G(s) = \frac{1}{s(s+1)}$) and let

$$D(s) = 70\frac{s+2}{s+10}e^{-Ts/2},$$

1. Obtain the overall closed loop transfer function between reference and output.
Closed-loop transfer function

$$H(s) = \frac{Y(s)}{R(s)} = \frac{D(s)G(s)}{1 + D(s)G(s)} = \frac{70(s+2)e^{-Ts/2}}{70(s+2)e^{-Ts/2} + s(s+1)(s+10)}.$$

2. Estimate the damping constant of the closed loop system when $T = 1/10,\ 1/20,\ 1/40$ both by hand calculation and by looking at simulation results (such as step responses). For hand calculations, you may use an approximation such as $e^{-sT/2} = \frac{2/T}{s+2/T}$.
Substituting $e^{-Ts/2} \simeq \frac{2/T}{s+2/T}$ and simplifying, we get

$$H(s) = \frac{70(s+2)}{\frac{T}{2}s^4 + \frac{T}{2}(11 + \frac{2}{T})s^3 + \frac{T}{2}(10 + \frac{22}{T})s^2 + 80s + 140}.$$

For $T = 1/10$,

$$H(s) = \frac{70(s+2)}{0.05s^4 + 1.55s^3 + 11.5s^2 + 80s + 140}.$$

After factoring the denominator, we get

$$H(s) = \frac{70(s+2)}{(s+23.99)(s+2.2873)(s^2 + 4.7228s + 51.0267)}.$$

The pole at $s = -23.99$ is farthest to the left of origin, and therefore will have minimum effect on the overall response. So, we can ignore this pole. The zero at $s = -2$ approximately eliminates the effect of the pole at $s = -2.2873$, resulting in the approximate transfer function

$$H(s) \simeq \frac{70}{s^2 + 4.7228s + 51.0267}.$$

Converting the transfer function to the form

$$A\frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2},$$

we get

$$1.37\frac{7.14^2}{s^2 + 2(0.33)(7.14)s + 7.14^2}.$$

From which we can see that $\zeta = 0.33$.
In Fig. 5, we can observe that the step response of the system has an overshoot of 0.459. From ζ-overshoot relationship,

$$e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.459,$$

$$\Rightarrow \zeta \simeq 0.24.$$

For $T = 1/20$,

$$H(s) = \frac{70(s+2)}{0.025s^4 + 1.275s^3 + 11.25s^2 + 80s + 140}.$$

After factoring the denominator, we get

$$H(s) = \frac{70(s+2)}{(s+42.03)(s+2.3148)(s^2 + 6.656s + 57.56)}.$$

Approximating the transfer function as done previously, we get

$$H(s) \simeq \frac{70}{s^2 + 6.656s + 57.56} \simeq 1.22\frac{7.59^2}{s^2 + 2(0.44)(7.59)s + 7.59^2}.$$

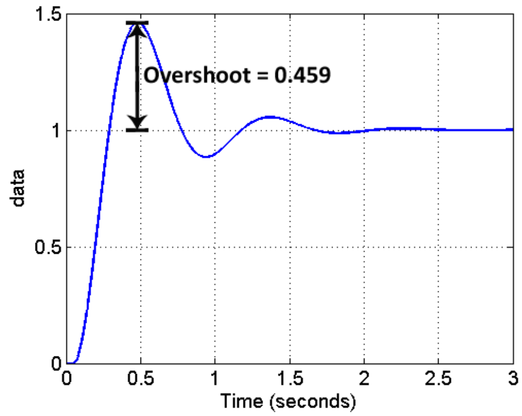

Figure 5: Step response with $T = 1/10$. Overshoot is also labelled.

From which we can see that $\zeta = 0.44$.
The step response of the system has an overshoot of 0.323. From ζ-overshoot relationship,

$$e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.323,$$

$$\Rightarrow \zeta \simeq 0.34.$$

For $T = 1/40$;

$$H(s) = \frac{70(s+2)}{0.0125s^4 + 1.138s^3 + 11.13s^2 + 80s + 140}.$$

After factoring the denominator, we get

$$H(s) = \frac{70(s+2)}{(s+81)(s+2.33)(s^2 + 7.71s + 59.34)}.$$

Approximating the transfer function as done previously, we get

$$H(s) \simeq \frac{70}{s^2 + 7.71s + 59.34} \simeq 1.18\frac{7.7^2}{s^2 + 2(0.5)(7.7)s + 7.7^2}.$$

From which we can see that $\zeta = 0.5$.
The step response of the system has an overshoot of 0.268. From ζ-overshoot relationship,

$$e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.268,$$

$$\Rightarrow \zeta \simeq 0.387.$$

3. If $e^{-sT/2}$ models the delay caused by ZOH, what are your conclusions from the above calculations?
We can conclude that the damping effect reduces for larger delays caused by ZOH.

- **Problem 5:**
  A computer disc drive is modeled by the plant

$$G(s) = \frac{1000}{s^2}.$$

1. Design and simulate a digital PID controller to meet a bandwidth of 100 Hz, phase margin of 50 degrees and zero error against step input torque. Use a sampling rate of 6 kHz for your design.

   The transfer function of a PID controller can also be written as

$$D(s) = \frac{K}{s}\Big((T_D s + 1)(s + \frac{1}{T_I})\Big),$$

   where K, T_D and T_I are proportional, derivative and integral parameters respectively. Now,

$$D(s)G(s) = \frac{1000K}{s^3}\Big((T_D s + 1)(s + \frac{1}{T_I})\Big).$$

   To compute the steady-state error e_{ss} at step input torque,

$$e_{ss} = \frac{1}{1 + \lim_{s\to 0}(D(s)G(s))} = \frac{1}{1+\infty} = 0,$$

   which shows that the steady-state error is already zero.

   Now for BW = 100, $\omega_c \simeq \frac{\omega_{BW}}{1.9} \simeq 52.6$ rad s^{-1}. From the Bode plots for $G(s)$ and PID controller as shown in Fig. 6, we can deduce that to get a PM of 50° at $\omega = 52.6$, we need

$$\omega_c T_I \simeq 30,$$
$$\Rightarrow T_I = 0.57.$$

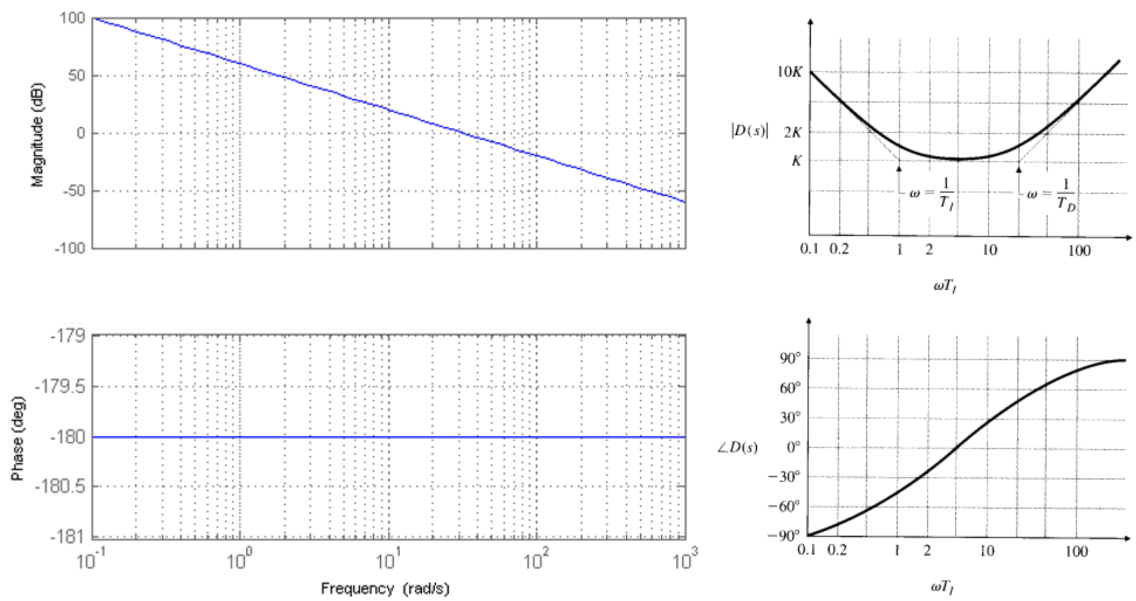

Figure 6: Frequency response of $G(s)$ (left) and frequency response of PID compensation (right) with $T_I/T_D = 20$.

   Since $\frac{T_I}{T_D} = 20 \Rightarrow T_D = 0.0285$.
   Now drawing the Bode plot of $D(s)G(s)$ with K = 1, we can see that to make $\omega_c = 52.6$, K should be 1.53. Finally we get,

$$D(s) = \frac{1.53}{s}\Big((0.0285s + 1)(s + 1.754)\Big).$$

Now, using Tustin's method for continuous to discrete transform, with $T = \dfrac{1}{f_s} = \dfrac{1}{6 \times 10^3}$,

$$D(z) = \frac{513.6z^2 - 1024z + 510.4}{z^2 - 1}.$$

Fig. 7 shows the Bode-plot of the closed-loop system, and Fig. 8 shows the closed-loop step response after applying the digital controller.

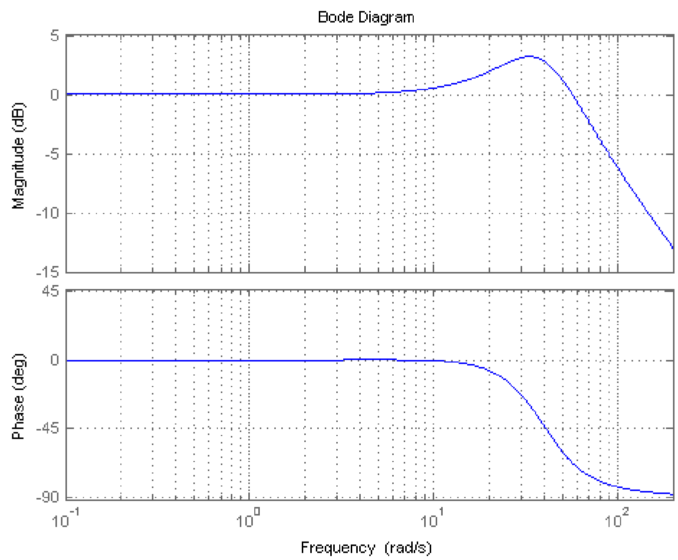

Figure 7: Bode plot of the closed-loop system.

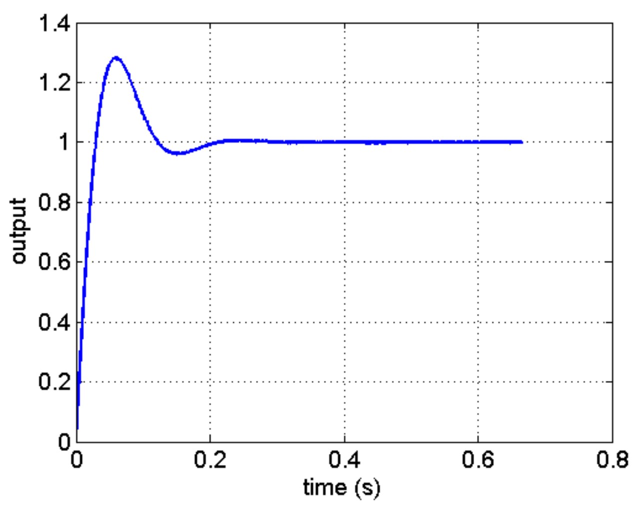

Figure 8: Step response of the close-loop system with the designed digital PID.

2. Analyze (in simulation) the change in PM (phase margin) due to the introduction of digital control by varying sampling rate.

   From the continuous-time controller, PM = 54.1°. For small sampling rates, PM is far from 54.1. For example, at 30 Hz, PM = 64°. But as the sampling rate decreases, PM converges to 54.1°.