

Kinematics and Dynamics of Mobile Robots



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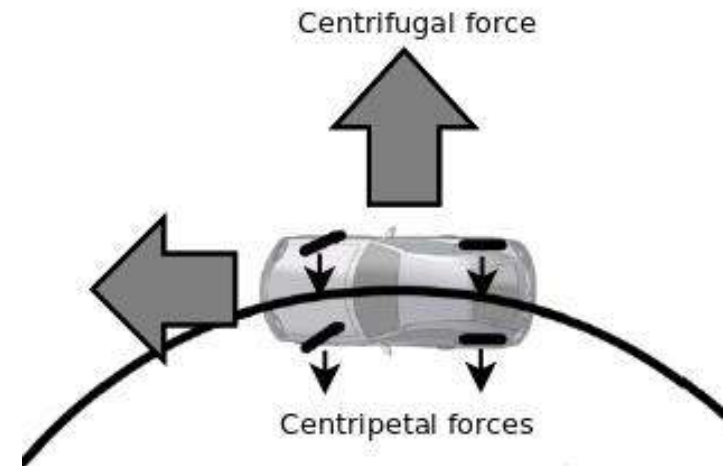
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Why Regarding Dynamics?

- Kinematics do not consider any forces applied to the system
- Control solely based on kinematics only suitable if forces are small and can be neglected
- At higher speeds or instable systems dynamic forces are often crucial
 - Vehicles (at higher velocities)
 - Walking machines
 - ...



Kinematics vs. Dynamics

- **Kinematics** - motion of bodies and systems of bodies without consideration of the masses nor the forces

- **Dynamics** - relationship between motion of bodies and their causes based on
 - Inertia
 - Elasticity
 - Friction

- **Application:**
 - Analysis
 - Synthesis of mechanics
 - Control design

Dynamics – Equation of Motion

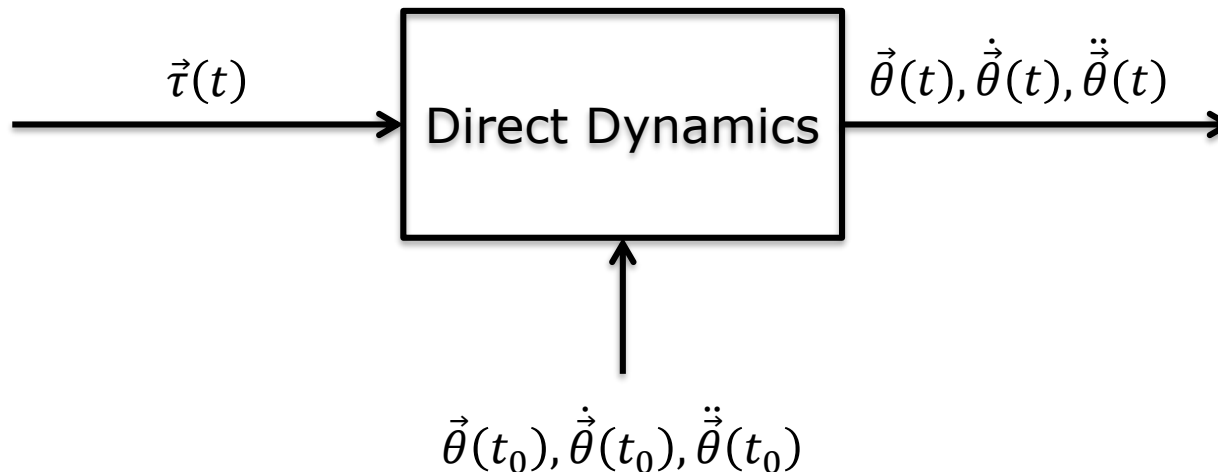
- Relationship between forces/moments, acceleration, velocity and position of objects

$$\vec{\tau} = I(\vec{\theta}) \cdot \ddot{\vec{\theta}} + n(\dot{\vec{\theta}}, \vec{\theta}) + g(\vec{\theta}) + R \cdot \dot{\vec{\theta}} \quad (1)$$

- $\vec{\tau}: n \times 1$ vector of general forces and moments
- $I(\vec{\theta}): n \times n$ moment of inertia
- $n(\dot{\vec{\theta}}, \vec{\theta}): n \times 1$ centrifugal and coriolis force
- $g(\vec{\theta}): n \times 1$ vector of gravitation component
- $R: n \times n$ diagonal-matrix describing friction
- $\vec{\theta}: n \times 1$ control variables

Direct Dynamics

- Calculate motion based on
 - external forces and moments
 - position
 - initial acceleration and velocity



Inverse Dynamics

- Calculate required forces and moments based on given motion

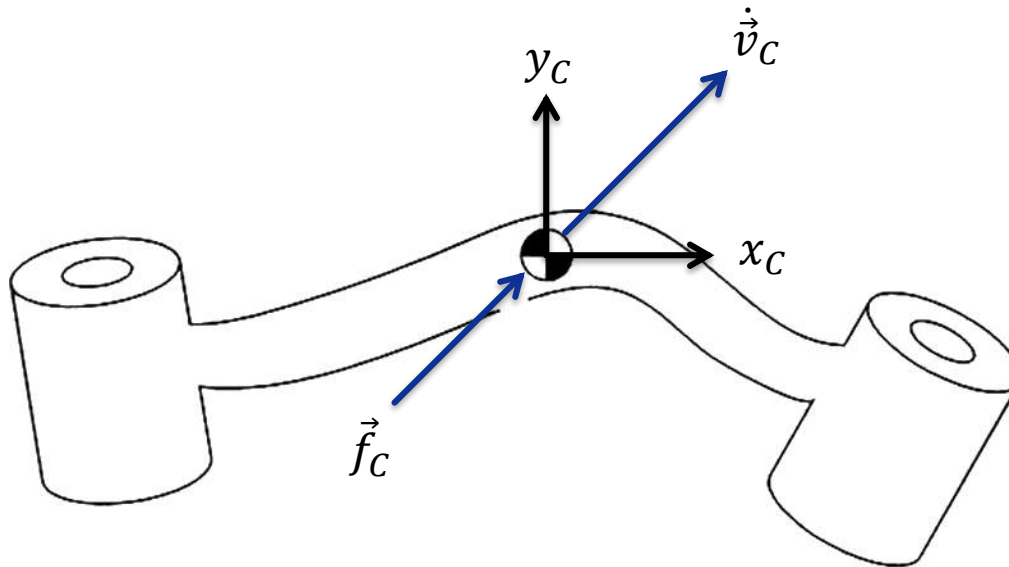


Derivation of the Dynamic Equations

- Synthetic Method (Newton-Euler):
Free-body diagram
 - Law of conservation of (angular-) momentum
 - Constraint force elimination leads to motion compensation
 - Basis for more complicated multi-body modes describing systems of rigid bodies connected by joints
- Analytic Method (Lagrange):
Application of the optimization principle
 - Energy consideration
 - Formal derivation yields motion equations

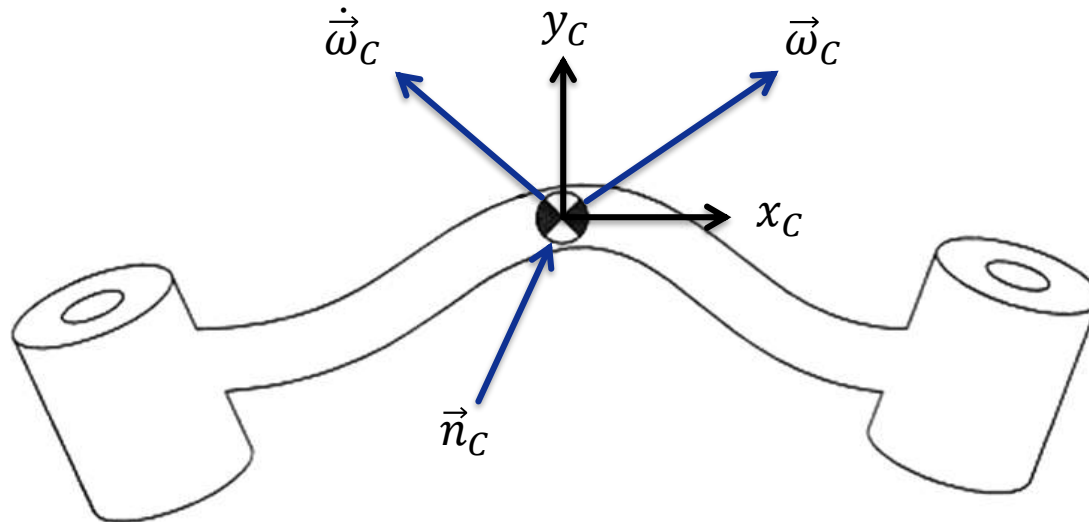
Newton-Euler: Base Equations

- Newton: $\vec{f}_C = m \cdot \dot{\vec{v}}_C$ (11)
 - m : overall mass
 - $\dot{\vec{v}}_C$: acceleration of the center of gravity C
 - \vec{f}_C : forces applying in C



Newton-Euler: Base Equations

- Euler: $\vec{n}_C = {}^C I \cdot \dot{\vec{\omega}}_C + \vec{\omega}_C \times {}^C I \cdot \vec{\omega}_C$ (12)
 - $\vec{\omega}_C$: angular velocity of the link
 - ${}^C I$: moment of inertia tensor in Frame C
 - \vec{n}_C : moment applying at center of mass which causes the rotation



Lagrange

- Motion equation according to Lagrange

$$\tau = \frac{d}{dt} \frac{\delta l}{\delta \dot{\theta}} - \frac{\delta l}{\delta \theta} \quad (13)$$

- θ : rotation angle or translation path
 - $\dot{\theta}$: object velocity
 - τ : vector of force/moment applying to the object
-
- Lagrange function: $l = e_{kin} - e_{pot}$ (with regard to the base) (14)
 - Describes the difference between kinetic and potential energy of a mechanical system

Lagrange: Kinetic Energy

- Kinetic Energy

$$e_{kin} = \underbrace{\frac{1}{2} m \cdot \vec{v}_C^T \cdot \vec{v}_C}_{\text{Linear Part}} + \underbrace{\frac{1}{2} \vec{\omega}^T \cdot C_I \cdot \vec{\omega}}_{\text{Rotational Part}} \quad (15)$$

- \vec{v}_C and $\vec{\omega}$ depend on position and velocity of the center of gravity

Lagrange: Kinetic Energy

- Kinetic energy depends on position and velocity

$$e_{kin}(\vec{\theta}, \dot{\vec{\theta}}) = \frac{1}{2} \dot{\vec{\theta}}^T \cdot M(\vec{\theta}) \cdot \dot{\vec{\theta}} \quad (16)$$

- $M(\vec{\theta})$: $n \times n$ mass-matrix, where each element is a complex function of $\vec{\theta}$
- $M(\vec{\theta})$: positiv-definite, i.e. $\dot{\vec{\theta}}^T \cdot M(\vec{\theta}) \cdot \dot{\vec{\theta}}$ always yields a positive scalar
- Equation corresponds to the common equation of kinetic energy of a point-mass

$$e_{kin} = \frac{1}{2} m \cdot v^2 \quad (17)$$

Lagrange: Potential Energy

- Potential Energy

$$e_{pot} = -m \cdot {}^0\vec{g}^T \cdot {}^0\vec{u}_C + e_{pot,ref} \quad (18)$$

- ${}^0\vec{g}$: 3×1 gravity vector, with regard to the origin frame 0
 - ${}^0\vec{u}_C$: 3×1 vector to the center of mass
 - $e_{pot,ref}$: constant, such that $e_{pot} \geq 0$ holds
-
- Potential energy can be defined as function $e_{pot}(\vec{\theta})$ based on the joint angle

Lagrange: Modeling of Nonholonomic Robots

- Lagrange dynamic model of a nonholonomic robot has the form

$$E\tau = \frac{d}{dt} \frac{\delta l}{\delta \dot{\theta}} - \frac{\delta l}{\delta \theta} + D^T(\theta)\lambda \quad (19)$$

- $D(\theta)$ is the $m \times n$ matrix of the m constraints introduced in the kinematic part

$$D(\theta)\dot{\theta} = 0$$

- λ is the vector Lagrange multiplier
- E is a non-singular transformation matrix
- This model leads to the motion equation

$$E\tau = I(\theta) \cdot \ddot{\theta} + n(\dot{\theta}, \theta) \cdot \dot{\theta} + g(\theta) + D^T(\theta)\lambda \quad (20)$$

Lagrange: Modeling of Nonholonomic Robots

- Elimination of constraint term by matrix $B(\theta)$ with

$$B^T(\theta)D^T(\theta) = 0 \quad (21)$$

- Due to the constraints we know that $v(t)$ exists with

$$\dot{\theta}(t) = B(\theta)v(t) \quad (22)$$

- Multiplying the motion equation by B^T

$$\bar{E}\tau = \bar{I}(\theta) \cdot \dot{v} + \bar{n}(\dot{\theta}, \theta) \cdot v + \bar{g}(\theta) \quad (23)$$

$$\bar{I} = B^T I B \quad (24)$$

$$\bar{n} = B^T I \dot{B} + B^T n B \quad (25)$$

$$\bar{g} = B^T g \quad (26)$$

$$\bar{E} = B^T E \quad (27)$$

Differential Drive Dynamics (Newton-Euler)

- Translational motion

$$F = m \cdot \dot{v} \quad (28)$$

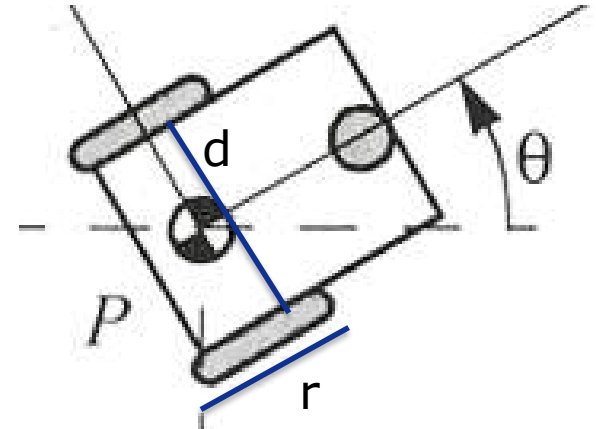
- Rotational motion

$$N = i \cdot \dot{\omega} \quad (29)$$

- Assuming CoG coincides with P

$$F = F_r + F_l = \frac{1}{r} (\tau_r + \tau_l) \quad (30)$$

$$N = (F_r - F_l) \cdot 2d = \frac{2d}{r} (\tau_r - \tau_l) \quad (31)$$



$$\tau_r = r \cdot F_r \quad (32)$$

$$\tau_l = r \cdot F_l \quad (33)$$

- Dynamic model

$$\dot{\xi}_W = R(\theta)^{-1} \dot{\xi}_r \quad (34)$$

$$\dot{\xi}_r = \begin{pmatrix} \dot{v} \\ 0 \\ \dot{\omega} \end{pmatrix} \quad (35)$$

$$\dot{v} = \frac{1}{mr} (\tau_r + \tau_l) \quad (36)$$

$$\dot{\omega} = \frac{2d}{ir} (\tau_r - \tau_l) \quad (37)$$

Differential Drive Dynamics (Lagrange)

- Assuming horizontal planar terrain and no slippage

$$g(\vec{\theta}) = 0 \quad n(\dot{\vec{\theta}}, \vec{\theta}) = 0 \quad (38)$$

- Lagrange motion model

$$E\tau = I(\theta) \cdot \ddot{\theta} + D^T(\theta)\lambda \quad (39)$$

- $$I(\theta) = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & i \end{bmatrix}; E = \frac{1}{r} \begin{bmatrix} \cos\omega & \cos\omega \\ \sin\omega & \sin\omega \\ 2d & -2d \end{bmatrix}$$

- $$B(\theta) = \begin{bmatrix} \cos\omega & 0 \\ \sin\omega & 0 \\ 0 & i \end{bmatrix}$$

- $$\bar{I} = B^T I B = \begin{bmatrix} m & 0 \\ 0 & i \end{bmatrix}; \bar{E} = \frac{1}{r} \begin{bmatrix} 1 & 1 \\ 2d & 2d \end{bmatrix}$$

Differential Drive Dynamics (Lagrange)

- Dynamic model becomes

$$\begin{bmatrix} m & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} = \frac{1}{r} \begin{bmatrix} 1 & 1 \\ 2d & -2d \end{bmatrix} \begin{bmatrix} \tau_r \\ \tau_l \end{bmatrix} \quad (40)$$

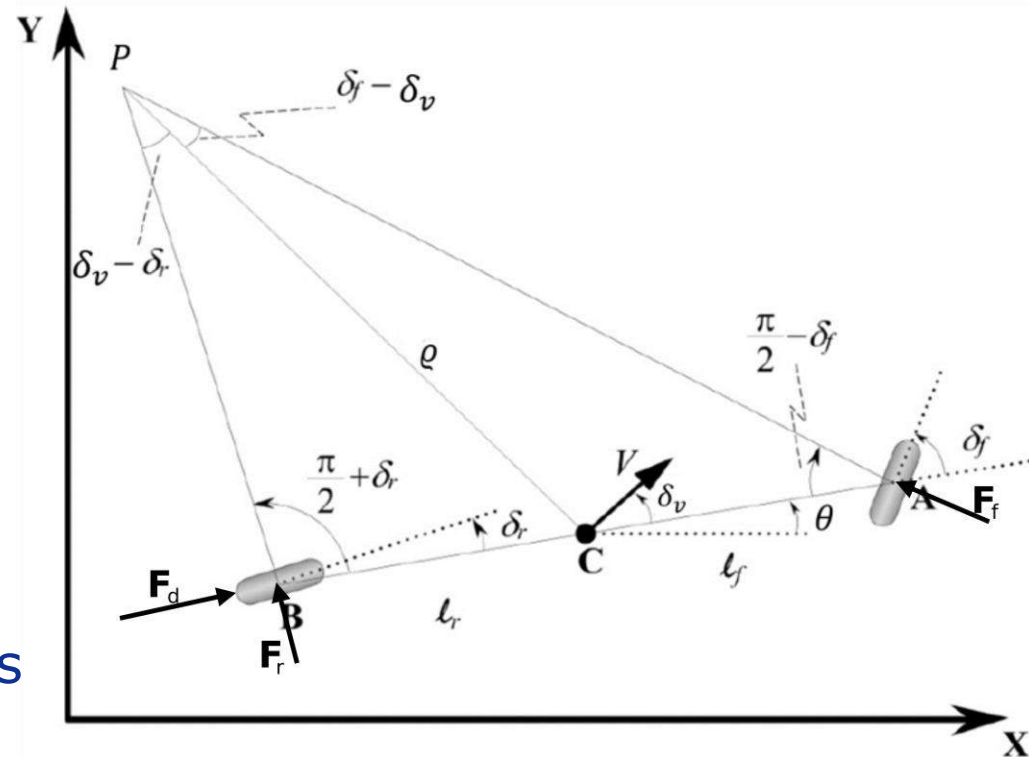
- Identical to model by Newton-Euler method

$$\dot{v} = \frac{1}{mr} (\tau_r + \tau_l) \quad \dot{\omega} = \frac{2d}{ir} (\tau_r - \tau_l) \quad (41)$$

Ackermann Dynamics

- Bicycle model
 - F_d driving force
 - F_r, F_f lateral slip forces
 - C center of gravity
 - $\delta_r = 0$

- Nonholonomic constraints according to kinematic
 - $\dot{x}_B \sin(\theta) - \dot{y}_B \cos(\theta) = 0$ (42)
 - $\dot{x}_A \sin(\theta + \delta_f) - \dot{y}_A \cos(\theta + \delta_f) = 0$ (43)



Ackermann Dynamics

- Need to express constraints in terms of center of gravity
- Wheel coordinates in terms of center of gravity

$$\begin{aligned}
 x_B &= x_C - l_r \cos(\theta) & x_A &= x_C + l_f \cos(\theta) \\
 y_B &= x_C - l_r \sin(\theta) & y_A &= x_C + l_f \sin(\theta) \\
 \dot{x}_B &= \dot{x}_C + l_r \dot{\theta} \sin(\theta) & \dot{x}_A &= \dot{x}_C - l_f \dot{\theta} \sin(\theta) \\
 \dot{y}_B &= \dot{y}_C - l_r \dot{\theta} \cos(\theta) & \dot{y}_A &= \dot{y}_C + l_f \dot{\theta} \cos(\theta)
 \end{aligned} \tag{44}$$

- Constraint in terms of center of gravity

$$\dot{x}_C \sin(\theta) - \dot{y}_C \cos(\theta) + l_r \dot{\theta} = 0 \tag{45}$$

$$\dot{x}_C \sin(\theta + \delta_f) - \dot{y}_C \cos(\theta + \delta_f) - l_f \dot{\theta} \cos(\theta) = 0 \tag{46}$$

Ackermann Dynamics

- Applying rotational transformation

$$\begin{bmatrix} \dot{x}_c \\ \dot{y}_c \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \dot{x}_{cR} \\ \dot{y}_{cR} \end{bmatrix} \quad (47)$$

to the constraints yields ((47) in (45) and (46))

$$\begin{aligned} \dot{y}_c &= l_r \dot{\theta} \quad \text{and} \quad \dot{\theta} = \frac{tg\delta_f}{D} \dot{x}_c \\ \rightarrow \dot{y}_c &= \frac{l_r}{D} (tg\delta_f) \dot{x}_c \end{aligned} \quad (48)$$

in the robot coordinate system.

- Need to regard acceleration for Newton-Euler base equations
-> differentiation

$$\begin{aligned} \ddot{y}_c &= l_r \ddot{\theta} \quad \text{and} \quad \ddot{\theta} = \frac{tg\delta_f}{D} \ddot{x}_c + \frac{1}{D \cos^2(\delta_f)} \dot{x}_c \dot{\delta}_f \\ \rightarrow \ddot{y}_c &= \frac{l_r}{D} (tg\delta_f) \ddot{x}_c + \frac{l_r}{D \cos^2(\delta_f)} \dot{x}_c \dot{\delta}_f \end{aligned} \quad (49)$$

Ackermann Dynamics – Newton-Euler

- m = mass of the robot
- J = moment of inertia
- F_d = driving force
- F_f, F_r = wheel lateral force
- T = time constant of steering system
- u_s = steering control point
- K = const coefficient
- r = rear wheel radius
- τ_d = applied motor torque

$$m(\ddot{x}_c - \dot{y}_c \dot{\theta}) = F_d - F_f \sin(\delta_f) \quad (50)$$

$$m(\dot{y}_c - \dot{x}_c \dot{\theta}) = F_r - F_f \cos(\delta_f) \quad (51)$$

$$J\ddot{\theta} = l_f F_f \cos(\delta_f) - l_r F_r \quad (52)$$

$$F_d = \left(\frac{1}{r}\right) \tau_d \quad (53)$$

$$\dot{\delta}_f = -\frac{1}{T} \delta_f + \frac{K}{T} u_s \quad (54)$$

Ackermann Dynamics – State Space Form

- $x = [x_c \quad y_c \quad \theta \quad \dot{x}_c \quad \delta_f]^T$
- Solve (52) for F_r , with (49) in (51)

$$\begin{aligned} m(l_r \ddot{\theta} + \dot{x}_c \dot{\theta}) &= F_f \cos(\delta_f) + \frac{l_f}{l_r} F_f \cos(\delta_f) - \frac{J}{l_r} \ddot{\theta} \\ &= \frac{(l_f + l_r) F_f \cos(\delta_f) - J \ddot{\theta}}{l_r} \end{aligned}$$

$$\Leftrightarrow F_f = \left(\frac{ml_r^2 + J}{(l_f + l_r) \cos(\delta_f)} \right) \ddot{\theta} + \left(\frac{ml_r \dot{x}_c}{(l_f + l_r) \cos(\delta_f)} \right) \dot{\theta} \quad (55)$$

- (48a,b), (49b), (55) in (50)

$$\begin{aligned} \ddot{x}_c &= \frac{\dot{x}_c (ml_r^2 + J) \operatorname{tg} \delta_f}{a} \dot{\delta}_f + \frac{(l_f + l_r)^2 \cos^2(\delta_f)}{a} F_d \\ a &= \left(\cos^2(\delta_f) \right) \left(m(l_f + l_r)^2 + (ml_r^2 + J) \operatorname{tg}^2 \delta_f \right) \end{aligned}$$

Ackermann Dynamics – State Space Form

- (48a,b) in (47) -> affine system with two inputs (τ_d, u_s)

$$\dot{x}_c = \left\{ \cos(\theta) - \left(\frac{\delta_f}{(l_f + l_r)} \right) (tg\delta_f) \sin(\theta) \right\} \dot{x}_c$$

$$\dot{y}_c = \left\{ \sin(\theta) - \left(\frac{\delta_f}{(l_f + l_r)} \right) (tg\delta_f) \cos(\theta) \right\} \dot{x}_c$$

$$\dot{\theta} = \left\{ \left(\frac{1}{(l_f + l_r)} \right) (tg\delta_f) \right\} \dot{x}_c$$

$$\ddot{x}_c = \left(\frac{1}{a} \right) \{ (ml_r^2 + J)(tg\delta_f)\dot{\delta}_f \dot{x}_c \} + \left(\frac{1}{a} \right) \left((l_f + l_r)^2 \cos^2(\delta_f) \right) F_d$$

$$\dot{\delta}_f = -\frac{\delta_f}{T} + \frac{K}{T} u_s$$

$$F_d = \frac{\tau_d}{r}$$

Ackermann Dynamics – State Space Form

- Drift term

$$g_0(x) = \begin{bmatrix} \left\{ \cos(\theta) - \left(\frac{\delta_f}{(l_f + l_r)} \right) (tg\delta_f) \sin(\theta) \right\} \dot{x}_c \\ \left\{ \sin(\theta) - \left(\frac{\delta_f}{(l_f + l_r)} \right) (tg\delta_f) \cos(\theta) \right\} \dot{x}_c \\ \left\{ \left(\frac{1}{(l_f + l_r)} \right) (tg\delta_f) \right\} \dot{x}_c \\ \left(\frac{1}{a} \right) \{ (ml_r^2 + J) (tg\delta_f) \dot{\delta}_f \dot{x}_c \} \\ - \frac{\delta_f}{T} \end{bmatrix}$$

Ackermann Dynamics – State Space Form

- Two input fields

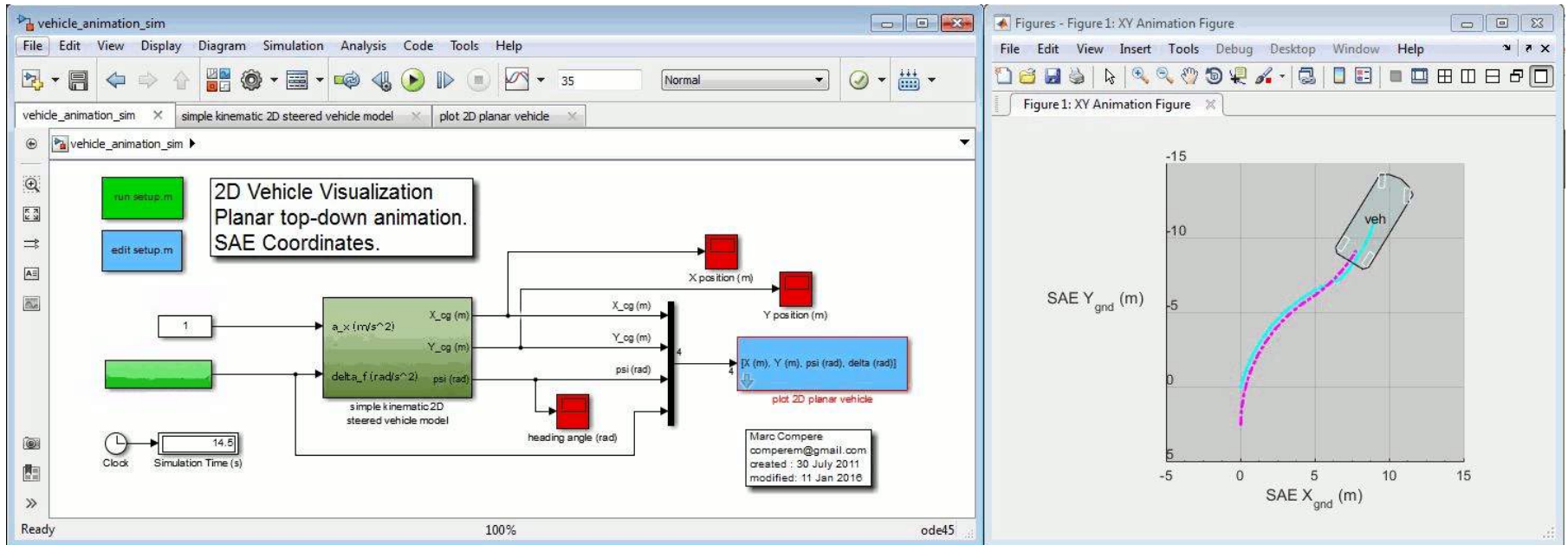
- $$g_1(x) = \left[0 \quad 0 \quad 0 \quad \frac{(l_f + l_r)^2 \cos^2(\delta_f)}{ra} \quad 0 \right]^T$$

- $$g_2(x) = \left[0 \quad 0 \quad 0 \quad 0 \quad \frac{K}{T} \right]^T$$

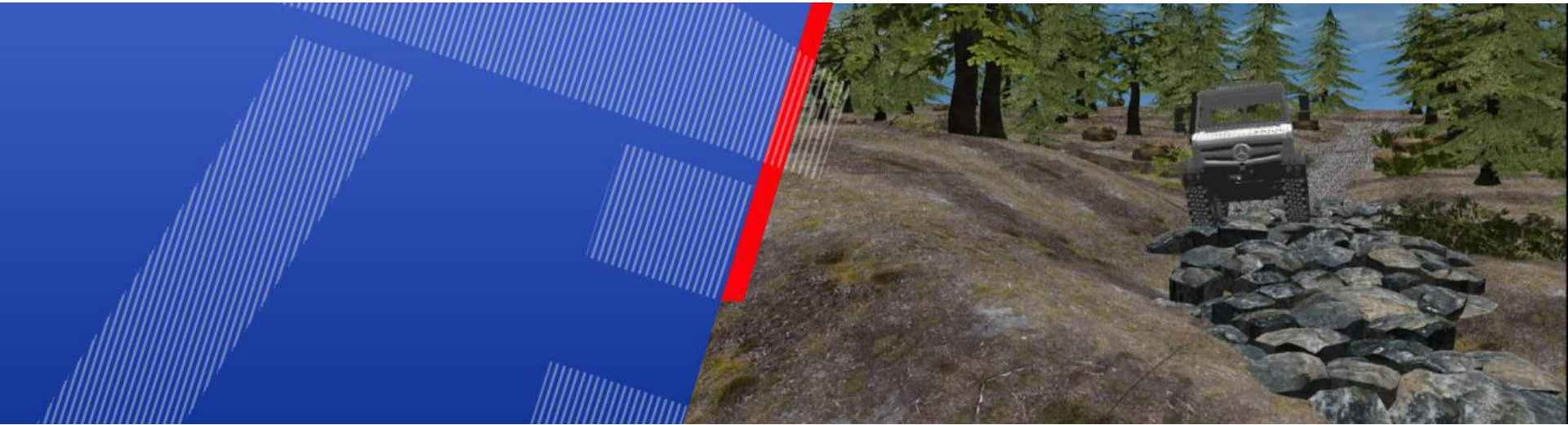
- Dynamics State Space Model

$$\dot{x} = g_0(x) + g_1(x)\tau_d + g_2(x)u_s$$

Ackermann Dynamics – Simulation



Thank you for your attention!



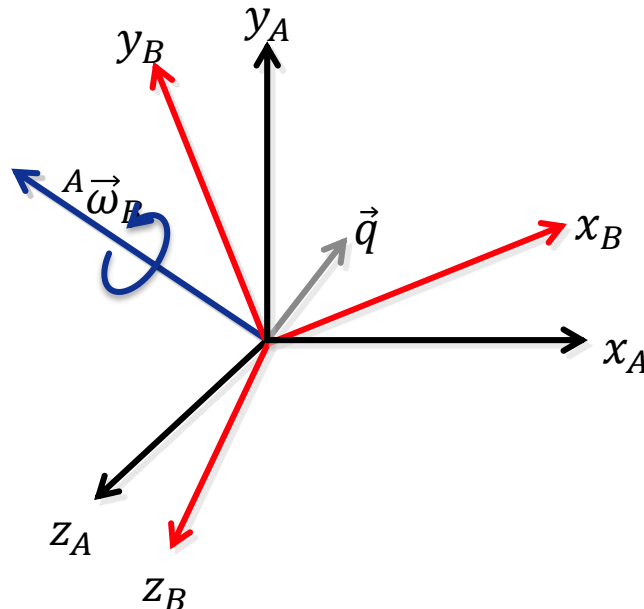
Acceleration of Rigid Bodies

- Linear Acceleration

$${}^B\dot{\vec{v}}_q = \frac{d}{dt} {}^B\vec{v}_q = \lim_{\Delta t \rightarrow 0} \frac{{}^B\vec{v}_q(t + \Delta t) - {}^B\vec{v}_q(t)}{\Delta t} \quad (2)$$

- Angular Acceleration

$${}^A\dot{\vec{\omega}}_B = \frac{d}{dt} {}^A\vec{\omega}_B = \lim_{\Delta t \rightarrow 0} \frac{{}^A\vec{\omega}_B(t + \Delta t) - {}^A\vec{\omega}_B(t)}{\Delta t} \quad (3)$$



Linear Acceleration

- Summary

$$\begin{aligned}
 {}^A\dot{\vec{v}}_q &= {}^A{}_B R \cdot {}^B\dot{\vec{v}}_q + 2 \cdot ({}^A\vec{\omega}_B \times {}^A{}_B R \cdot {}^B\vec{v}_q) \\
 &\quad + {}^A\dot{\vec{\omega}}_B \times {}^A{}_B R \cdot {}^B\vec{q} + {}^A\vec{\omega}_B \times ({}^A\vec{\omega}_B \times {}^A{}_B R \cdot {}^B\vec{q})
 \end{aligned} \tag{8}$$

- General Case (Frames A, B common origin)

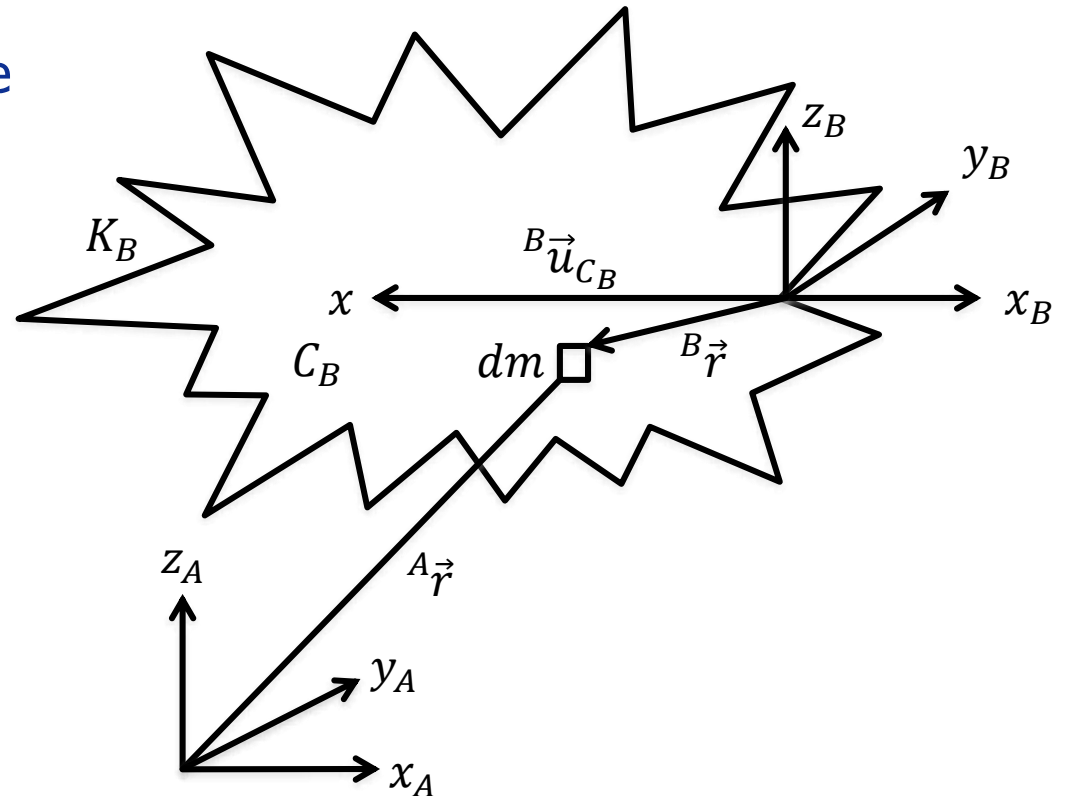
$$\begin{aligned}
 {}^A\dot{\vec{v}}_q &= {}^A\dot{\vec{v}}_{OB} + {}^A{}_B R \cdot {}^B\dot{\vec{v}}_q + 2 \cdot ({}^A\vec{\omega}_B \times {}^A{}_B R \cdot {}^B\vec{v}_q) \\
 &\quad + {}^A\dot{\vec{\omega}}_B \times {}^A{}_B R \cdot {}^B\vec{q} + {}^A\vec{\omega}_B \times ({}^A\vec{\omega}_B \times {}^A{}_B R \cdot {}^B\vec{q})
 \end{aligned} \tag{9}$$

- ${}^B\vec{q}$ not moving

$$\begin{aligned}
 {}^B\vec{v}_q &= {}^B\dot{\vec{v}}_q = \vec{0} \\
 \Rightarrow {}^A\dot{\vec{v}}_q &= {}^A\dot{\vec{v}}_{OB} + {}^A\dot{\vec{\omega}}_B \times {}^A{}_B R \cdot {}^B\vec{q} + {}^A\vec{\omega}_B \times ({}^A\vec{\omega}_B \times {}^A{}_B R \cdot {}^B\vec{q})
 \end{aligned} \tag{10}$$

Mass Distribution: Preliminary Considerations

- dm : Mass particle
- C_B : Center of Gravity of Body B
- \vec{u}_{C_B} : Vector to C_B
- \vec{r} : Vector to the particle



Mass Distribution: Moment of Inertia Tensor

- Moment of Inertia Tensor in Frame A describes the rotatory inertia of an object

$${}^A I = \begin{bmatrix} {}^A i_{XX} & -{}^A i_{xy} & -{}^A i_{xz} \\ -{}^A i_{xy} & {}^A i_{yy} & -{}^A i_{yz} \\ -{}^A i_{xz} & -{}^A i_{yz} & {}^A i_{zz} \end{bmatrix}$$

- Scalar elements given by integration over mass distribution M

- Axial moments of inertia

$${}^A i_{xx} = \iiint_M (y_A^2 + z_A^2) dm \quad {}^A i_{yy} = \iiint_M (x_A^2 + z_A^2) dm \quad {}^A i_{zz} = \iiint_M (x_A^2 + y_A^2) dm$$

- moments of inertia product

$${}^A i_{xy} = \iiint_M x_A y_A dm \quad {}^A i_{xz} = \iiint_M x_A z_A dm \quad {}^A i_{yz} = \iiint_M y_A z_A dm$$

- Moment of inertia tensor of a point mass is a zero matrix

Mass Distribution: Cuboid

- Moment of inertia of a cuboid of uniform density ρ

- $dm = \rho dx dy dz$ yields

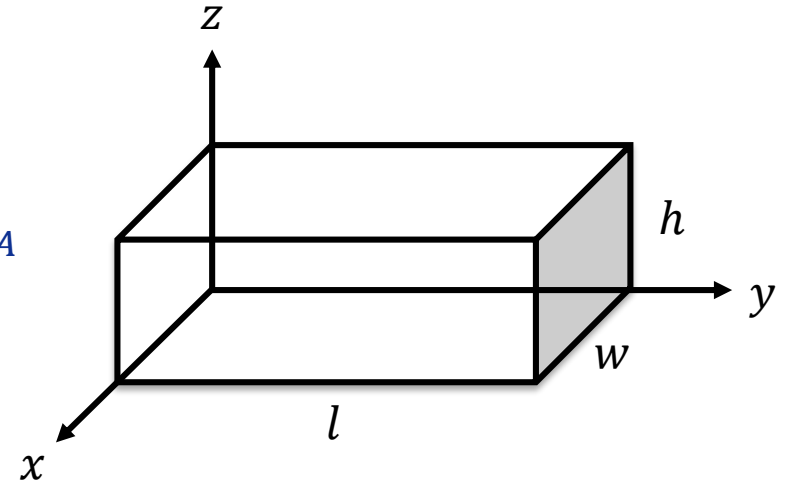
$${}^A i_{xx} = \int_0^h \int_0^l \int_0^w (y_A^2 + z_A^2) \rho dx_A dy_A dz_A$$

$$= \int_0^h \int_0^l (y_A^2 + z_A^2) w \rho dy_A dz_A$$

$$= \int_0^h \left(\frac{l^3}{3} + z_A^2 l \right) w \rho dz_A$$

$$= \left(\frac{hl^3 w}{3} + \frac{h^3 lw}{3} \right) \rho$$

$$= \frac{m}{3} (l^2 + h^2) \quad (\text{with overall mass } m)$$



- Analogously,

$${}^A i_{yy} = \frac{m}{3} (w^2 + h^2)$$

$${}^A i_{zz} = \frac{m}{3} (l^2 + w^2)$$