MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
2.151 Advanced System Dynamics and Control

Repeated and Complex Eigenvalues: "Almost" Diagonal Systems[1]

Introduction: We have seen that a linear system

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} &= \mathbf{Cx} + \mathbf{Du}\end{aligned}$$

with distinct eigenvalues can be transformed to a diagonal representation by the similarity transform

$$\begin{aligned}\dot{\mathbf{z}} &= \mathbf{\Lambda z} + \mathbf{M}^{-1}\mathbf{Bu} \\ \mathbf{y} &= \mathbf{CMz} + \mathbf{Du}\end{aligned} \qquad (1)$$

where $\mathbf{\Lambda} = \mathbf{M}^{-1}\mathbf{AM}$ is a diagonal matrix with the system eigenvalues on the diagonal, and $\mathbf{M}$ is the modal matrix, composed of columns of linearly independent eigenvectors. In this handout we address two separate issues: 1) the handling of systems with repeated eigenvalues when the eigenvectors are not necessarily linearly independent, and 2) the creation of an almost diagonal form when a system has complex conjugate eigenvalues and a purely real $\mathbf{A}$ matrix is desired for computational convenience.

Systems with Repeated Eigenvalues: If the matrix $\mathbf{A}$ has repeated eigenvalues it may or may not be possible to find a diagonal system representation in the form of Eq. (1). Consider the following example

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■ Example

Determine whether it is possible to transform the following two matrices to diagonal form

$$\text{(a)} \quad \mathbf{A}_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \text{(b)} \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution: Both matrices have identical eigenvalues $\lambda_{1,2,3} = 1, 1, 2$. For matrix $\mathbf{A}_1$ the modal matrix of associated eigenvectors is

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[1] D. Rowell 11/22/04

and we note that the columns are linearly independent (the matrix has rank 3). The transformed matrix is therefore

$$\mathbf{\Lambda}_1 = \mathbf{M}^{-1}\mathbf{A}_1\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

with the eigenvalues on the leading diagonal as expected.

For A_2 the situation is different. The eigenvectors associated with the repeated eigenvalue are not linearly independent and the modal matrix is

$$\mathbf{M}_2 = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\mathbf{M}_2$ is singular (rank of 2). The inverse $\mathbf{M}_2^{-1}$ does not exist and the transformation to diagonal form Using Eq. (1) cannot be found.

From this example we see that, if a square matrix has repeated eigenvalues, it is not always possible to find a set of linearly independent eigenvectors that form a basis, and consequently the matrix cannot always be transformed to diagonal form. It is, however, possible to find an alternative set of linearly independent basis vectors that allow transformation to an *almost diagonal* form, known as the *Jordan canonical* form. This representation has the system eigenvalues of A on the leading diagonal, and either 1 or 0 on the superdiagonal.

Consider a fifth-order system $\mathbf{A}$ with an eigenvalues λ_1 with multiplicity 4, and λ_2 with multiplicity 1. The Jordan form representation $\mathbf{J}$ of this system will have one of the following forms

$$\mathbf{J}_1 = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}, \mathbf{J}_2 = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}, \mathbf{J}_3 = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$
$$\mathbf{J}_4 = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}, \mathbf{J}_5 = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \tag{2}$$

Examination of these forms indicates that each matrix $\mathbf{J}_i$ is formed with one or more *blocks* associated with the repeated eigenvalue on the main diagonal, and that each of these blocks is of the form

$$[\lambda_1], \quad \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix}$$

with the repeated eigenvalue on the diagonal and 1's on the diagonal just above the main diagonal. These blocks are known as *Jordan blocks*. For example the matrix $\mathbf{J}_4$ above has two Jordan blocks associated with λ_1 (one of order 3 and one of order 1), and one Jordan block associated with λ_2. A diagonal matrix can be considered as a special case of the Jordan form in which all blocks are of order 1. Matrix $\mathbf{J}_1$ above can be considered to have four Jordan blocks of order 1 associated with eigenvalue λ_1, and a single Jordan block associated with eigenvalue λ_2. The Jordan form $\mathbf{J}$ can be considered a generalized form of the diagonal form $\mathbf{\Lambda}$.

The components of the $\mathbf{B}$ matrix associated with each Jordan block in the state equations

$$\dot{\mathbf{x}} = \mathbf{J}\mathbf{x} + \mathbf{B}\mathbf{u}. \tag{3}$$

will have the form

$$\mathbf{B} = \left[\begin{array}{ccccccc} \cdots & \vdots & 0 \quad 0 & \cdots & b_i & \vdots & \cdots \end{array} \right]^T \tag{4}$$

where b_i is a scalar quantity (usually 1). Thus a SISO system with Jordan form as in $\mathbf{J}_3$ above (with two second-order Jordan blocks) will have state equations

$$\dot{\mathbf{x}} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ b_2 \\ 0 \\ b_4 \\ b_5 \end{bmatrix} u. \tag{5}$$

Notice that each Jordan block represents a set of coupled equations, and that the block as a whole is uncoupled from the rest of the state equations, for example a third-order block associated with repeated eigenvalue λ will have state equations

$$\begin{aligned} \dot{x}_1 &= \lambda x_1 + x_2 \\ \dot{x}_2 &= \lambda x_1 + x_3 \\ \dot{x}_3 &= \lambda x_3 + b_3 u \end{aligned}$$

which represents a chain of cascaded first-order blocks, as shown in Fig. 1. A system with

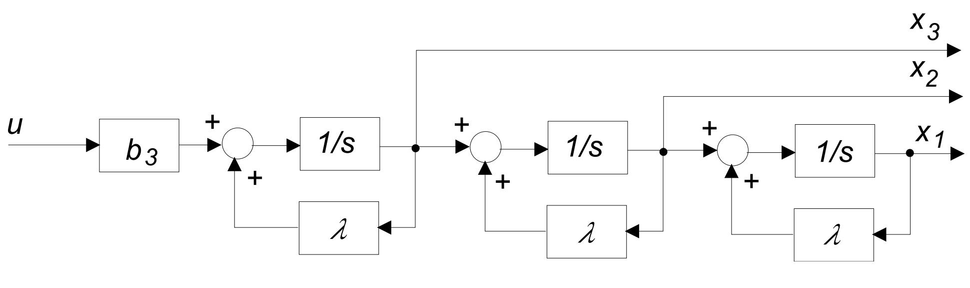

Figure 1: Bock diagram of a third-order Jordan block

multiple Jordan blocks is represented as a parallel combination of such chains. For example the system represented by Eq. (5) has a block diagram as shown in Fig. 2.

An important result can be stated as follows:

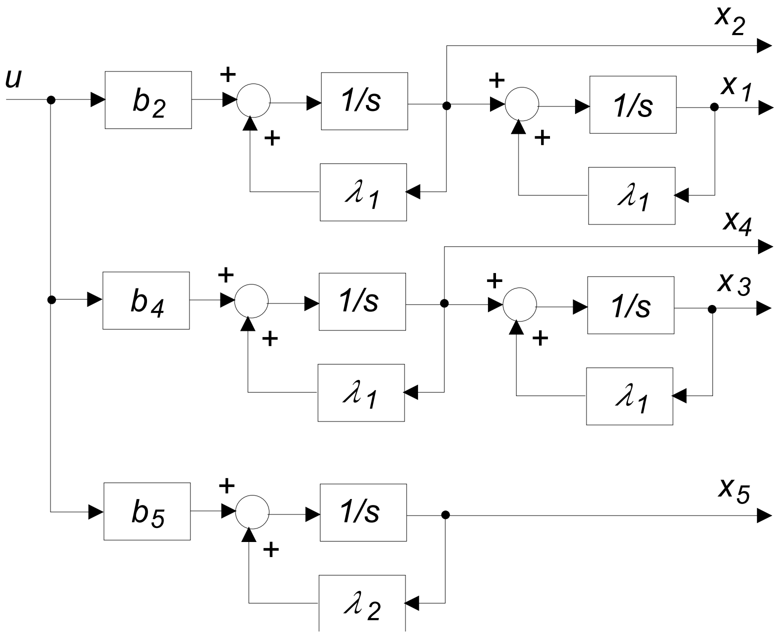

Figure 2: Block diagram of the fifth-order system with an eigenvalue of multiplicity four, and two linearly independent eigenvectors, described by Eq. (5).

Theorem 1: Any system in Jordan form, that is

$$\dot{\mathbf{x}} = \mathbf{J}\mathbf{x} + \mathbf{B}\mathbf{u}.$$

in which there is more than one Jordan block in $\mathbf{J}$ associated with any single eigenvalue is uncontrollable.

In the matrices $\mathbf{J}_1 \ldots \mathbf{J}_5$ in Eq. (8), only matrix $\mathbf{J}_5$ (a single Jordan block of order 4) will yield a controllable system. We also note that Theorem 1 states that the system described by Eq. (5), and shown in Fig. 2, is uncontrollable.

■ Example

Show that the second-order system represented by

$$\dot{\mathbf{x}} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \mathbf{x} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$

is uncontrollable (two first-order Jordan blocks representing the repeated eigenvalue $\lambda = -a$, while the system

$$\dot{\mathbf{x}} = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} u$$

(a single Jordan block of order two) is controllable.

For the first system, the controllability matrix $\mathbf{\Theta}_c$ is

$$\mathbf{\Theta}_c = [\mathbf{B} \,|\, \mathbf{AB}] = \begin{bmatrix} b_1 & ab_1 \\ b_2 & ab_2 \end{bmatrix}$$

which has rank = 1, and is therefore uncontrollable, whereas for the second system

$$\mathbf{\Theta}_c = [\mathbf{B} \,|\, \mathbf{AB}] = \begin{bmatrix} 0 & b_2 \\ b_2 & ab_2 \end{bmatrix}$$

which has rank = 2, indicating that this system is controllable.

Determination of the Jordan Form: When a system of order n has repeated eigenvalues, it may or may not be possible to find a set of linearly independent eigenvectors in the modal matrix $\mathbf{M}$ for the transformation to diagonal form. The number of linearly independent eigenvectors associated with an eigenvalue λ_i with multiplicity μ_i is equal to the *degeneracy* q_i of $\mathbf{A} - \lambda_i \mathbf{I}$, defined as

$$q_i = n - \text{rank}\,(\mathbf{A} - \lambda_i \mathbf{I}) \tag{6}$$

where $1 \leq q_i \leq \mu_i$. Under such conditions there will be q_i eigenvectors, and $m_i - q_i$ *generalized eigenvectors* associated with λ_i that form a linearly independent set of vectors for use as m_i columns of the matrix $\mathbf{M}$ to transform to a Jordan form.

A generalized eigenvector of rank k is defined as a non-zero vector $\mathbf{x}_k$ satisfying

$$\begin{aligned} & (\mathbf{A} - \lambda_i \mathbf{I})^k \, \mathbf{x}_k = 0 \\ \text{and} \qquad & (\mathbf{A} - \lambda_i \mathbf{I})^{k-1} \, \mathbf{x}_k \neq 0 \end{aligned} \tag{7}$$

Therefore if $\mathbf{x}_1$ is an eigenvector satisfying $(\mathbf{A} - \lambda_1 \mathbf{I})\,\mathbf{x}_1 = 0$, the generalized eigenvectors $\mathbf{x}_i$, $i = 2 \ldots \mu_i$, may be derived from the relationships

$$\begin{aligned} \mathbf{x}_1 &= (\mathbf{A} - \lambda_i \mathbf{I})\,\mathbf{x}_2 \\ \mathbf{x}_2 &= (\mathbf{A} - \lambda_i \mathbf{I})\,\mathbf{x}_3 \\ \vdots &= \vdots \\ \mathbf{x}_{\mu_{i-2}} &= (\mathbf{A} - \lambda_i \mathbf{I})\,\mathbf{x}_{\mu_{i-1}} \\ \mathbf{x}_{\mu_{i-1}} &= (\mathbf{A} - \lambda_i \mathbf{I})\,\mathbf{x}_{\mu_i}. \end{aligned} \tag{8}$$

Because $(\mathbf{A} - \lambda_i \mathbf{I})$ is singular, these equations cannot be solved directly, and each equation will lead to a set of relationships *between* the elements of the generalized eigenvector.

A simple (distinct) eigenvector may be found by either of the following methods, and used to seed the recursive calculation of the generalized eigenvectors:

(a) For a distinct eigenvalue, a non-trivial solution to the homogeneous equation

$$(\mathbf{A} - \lambda_i \mathbf{I})\,\mathbf{x}_i = \mathbf{0}$$

is given by any nonzero column of the matrix $\text{Adj}\,(\mathbf{A} - \lambda_i \mathbf{I})$, or

(b) The linear equations may be solved directly by substituting the numeric values of λ_i into $(\mathbf{A} - \lambda_i \mathbf{I})\,\mathbf{x}_i = \mathbf{0}$.

■ Example

Find the eigenvalues, eigenvectors, and if necessary the generalized eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -12 & -6 \end{bmatrix}$$

The characteristic equation is $(\lambda + 2)^3 = 0$ giving an eigenvalue $\lambda = -2$ with multiplicity $\mu = 3$. The rank of $(\mathbf{A} - \lambda_i \mathbf{I})\,|_{\lambda=-2}$ is 2, therefore the degeneracy $q = 3-2 = 1$. There is therefore one eigenvector and two generalized eigenvectors. Using method (a) above, the eigenvector is found from a nonzero column of

$$\mathrm{Adj}\,(\mathbf{A} + 2\mathbf{I}) = \begin{bmatrix} 1 & 1 & 1 \\ -8 & -8 & -2 \\ 16 & 16 & 4 \end{bmatrix}, \quad \text{and we may therefore select} \quad \mathbf{x}_1 = \begin{bmatrix} 1 & -2 & 4 \end{bmatrix}^T$$

The generalized eigenvectors may be found using $(\mathbf{A} + 2\mathbf{I})\mathbf{x}_2 = \mathbf{x}_1$ and $(\mathbf{A} + 2\mathbf{I})\mathbf{x}_3 = \mathbf{x}_2$. Note that the eigenvectors are not unique, and these equations will simply give relationships between the elements of the vector. For example, to find x_2:

$$(\mathbf{A} + 2\mathbf{I})\,\mathbf{x}_2 = \mathbf{x}_1 \quad \text{or} \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -8 & -12 & -4 \end{bmatrix} \begin{bmatrix} x_{2,1} \\ x_{2,2} \\ x_{2,3} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

from which $x_{2,1} = (1 - x_{2,2})/2$ and $x_{2,3} = -2 - 2x_{2,2}$. If we let $x_{2,2} = 1$ then the generalized eigenvector is $\mathbf{x}_2 = \begin{bmatrix} 0 & 1 & -4 \end{bmatrix}^T$. Similarly, we can find $\mathbf{x}_3 = \begin{bmatrix} -1/2 & 1 & -1 \end{bmatrix}^T$. The transformation matrix $\mathbf{M}$ is

$$\mathbf{M} = \begin{bmatrix} \mathbf{x}_1 & \vdots & \mathbf{x}_2 & \vdots & \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1/2 \\ -2 & 1 & 1 \\ 4 & -4 & -1 \end{bmatrix}$$

and the Jordan form consists of a single third-order block:

$$\mathbf{J} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

The Jordan Form and the Transfer Function: An alternative approach to finding a Jordan form for a controllable system is based on the transfer function. Consider a single-input single-output system with an nth order transfer function $H(s)$, and eigenvalues $\lambda_1, \lambda_2 \ldots \lambda_n$

$$\begin{aligned} H(s) &= \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D} \\ &= \frac{N(s)}{(s-\lambda_1)(s-\lambda_2)\ldots(s-\lambda_n)} \end{aligned} \tag{9}$$

If the eigenvalues are distinct, $H(s)$ may be expressed in partial fractions:

$$H(s) = \frac{c_1}{(s-\lambda_1)} + \frac{c_2}{(s-\lambda_2)} + \ldots + \frac{c_n}{(s-\lambda_n)}, \tag{10}$$

where the constants c_i are the residues at the poles λ_i. Then the system output is

$$Y(s) = c_1 \frac{U(s)}{(s-\lambda_1)} + c_2 \frac{U(s)}{(s-\lambda_2)} + \ldots + c_n \frac{U(s)}{(s-\lambda_n)} \tag{11}$$

which is a parallel representation based on *uncoupled first-order* blocks. If we assign state variables x_i as the output of each such block, the set of state and output equations may be written

$$\begin{aligned}
\dot{x}_1 &= \lambda x_1 + u \\
\dot{x}_2 &= \lambda x_2 + u \\
\vdots &= \vdots \\
\dot{x}_n &= \lambda x_n + u \\
y &= c_1 x_1 + c_2 x_2 \ldots + c_n x_n
\end{aligned}$$

or

$$\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} c_1 & c_2 \ldots & c_n \end{bmatrix} \mathbf{x}
\end{aligned} \tag{12}$$

which is clearly in diagonal form, as shown in Fig. 3.

When the system contains repeated eigenvalues, the simple partial fraction expansion of Eq. (10) no longer applies. Instead, the terms involving a pole λ_i of multiplicity m must be written as

$$\begin{aligned}
H(s) &= \frac{N(s)}{(s-\lambda_1)\ldots(s-\lambda_i)^m\ldots(s-\lambda_n)} \\
&= \frac{c_1}{(s-\lambda_1)} + \ldots + \left(\frac{c_i}{(s-\lambda_i)} + \frac{c_{i+1}}{(s-\lambda_i)^2} + \ldots + \frac{c_{i+m-1}}{(s-\lambda_i)^m} \right) + \ldots + \frac{c_n}{(s-\lambda_n)}.
\end{aligned}$$

The m terms associated with higher order terms $1/(s-\lambda_i)^k$ may be represented by a cascaded chain of m first-order blocks, for example Fig. 4 shows the block diagram representation of a fourth-order transfer function with a repeated eigenvalue λ_2 with a multiplicity 2. The state equation representation is shown in Fig. 4(b), and the state equations are

$$\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} \mathbf{x}
\end{aligned}$$

which is in Jordan form, with a second-order Jordan block associated with λ_2.

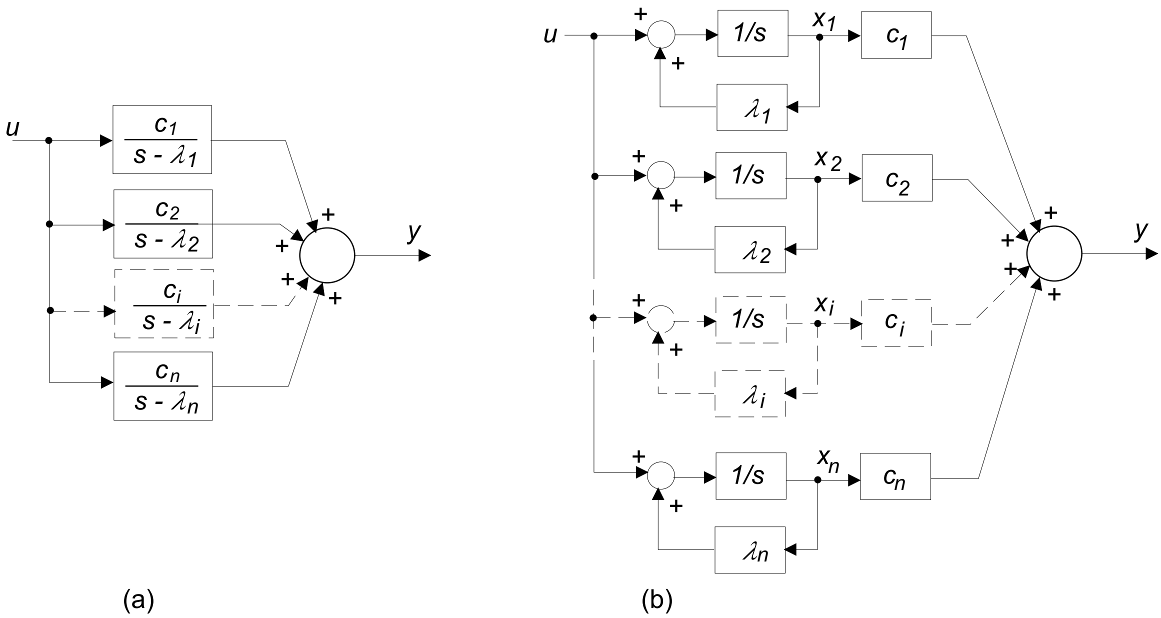

Figure 3: Diagonal form for a system with distinct eigenvalues (a) derived from partial fraction expansion of the transfer function, and (b) the block diagram for the equivalent state equations.

■ Example

Derive a Jordan form representation of the system with transfer function

$$H(s) = \frac{3s^2 + 30s + 72}{s^5 + 12s^4 + 53s^3 + 110s^2 + 108s + 40}$$

The characteristic equation is $(s+1)(s+3)^3(s+5) = 0$, giving eigenvalues $\lambda_{1\ldots5} = -1, -3, -3, -3, -5$. The partial fraction expansion of $H(s)$ is

$$H(s) = \frac{45/4}{s+1} - \frac{8}{(s+2)^3} - \frac{34/3}{(s+2)^2} - \frac{101/9}{(s+2)} + \frac{1/36}{s+5}$$

Let

$$\begin{aligned} X_4(s) &= \frac{1}{s+2}U(s) \\ X_3(s) &= \frac{1}{(s+2)^2}U(s) = \frac{1}{s+2}X_4(s) \\ X_2(s) &= \frac{1}{(s+2)^3}U(s) = \frac{1}{s+2}X_3(s), \end{aligned}$$

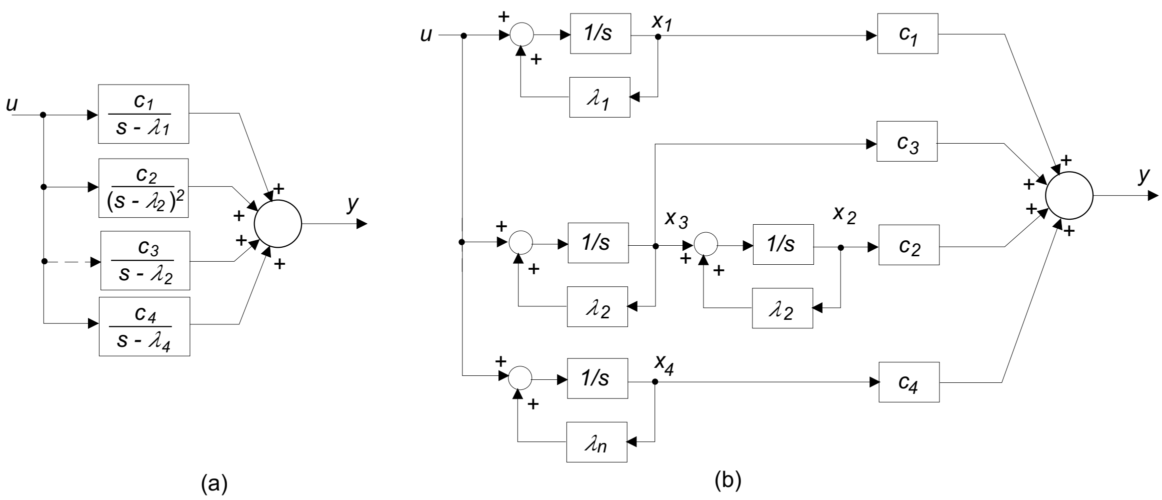

Figure 4: Diagonal form of a system with a repeated eigenvalue: (a) the block diagram from the partial fraction expansion of the transfer function, and (b) the block diagram of the equivalent Jordan form state equations.

then the state equations can be written by inspection

$$
\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} 55/4 & -8 & -34/3 & -101/9 & -1/36 \end{bmatrix} \mathbf{x}
\end{aligned}
$$

which contain a third-order Jordan block representing the three state equations

$$
\begin{aligned}
\dot{x}_2 &= -\lambda_2 x_2 + x_3 \\
\dot{x}_3 &= -\lambda_2 x_3 + x_4 \\
\dot{x}_4 &= -\lambda_2 x_4 + u
\end{aligned}
$$

It should be noted that the transfer function describes only the completely controllable and observable subsystem of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$. Therefore an uncontrollable or unobservable system will not be described completely by a Jordan form derived from the transfer function. In addition, because the transfer function is controllable, Theorem 1 guarantees that there will be at most a single Jordan block in any system represented by a transfer function.

Real System Representation for Complex Eigenvalues: If a system with distinct eigenvalues has one or more pairs of complex conjugate eigenvalue pairs, the transformed diagonal form $\mathbf{\Lambda}$ will have an $\mathbf{A}$ matrix with complex conjugate elements on the diagonal, for example

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \sigma + j\omega & & & \\ & & & \sigma - j\omega & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{bmatrix} \tag{13}$$

where the pair of eigenvalues $\lambda_{i,i+1} = \sigma \pm j\omega$. We note that the corresponding eigenvectors are also complex conjugates.

■ Example

The system

$$H(s) = \frac{2}{(s+1)(s^2+2s+5)}$$

in phase-variable form is represented by

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -7 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \mathbf{x} \end{aligned}$$

with eigenvalues $\lambda_{1,2,3} = -1, -1+j2, -1-j2$. A corresponding modal matrix, and its inverse, are

$$\begin{aligned} \mathbf{M} &= \begin{bmatrix} 0.5774 & -0.1078 + j0.1437 & -0.1078 - j0.1437 \\ -0.5774 & -0.1796 - j0.3592 & -0.1796 + j0.3592 \\ 0.5774 & 0.8980 & 0.8980 \end{bmatrix} \\ \mathbf{M}^{-1} &= \begin{bmatrix} 2.1651 & 0.8660 & 0.4330 \\ -0.6960 + j1.3919 & -0.2784 + j1.9487 & 0.4176 + j0.5568 \\ -0.6960 - j1.3919 & -0.2784 - j1.9487 & 0.4176 - j0.5568 \end{bmatrix} \end{aligned}$$

and transformation according to Eqs. (1) gives the diagonal system

$$\begin{aligned} \dot{\mathbf{z}} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1+j2 & 0 \\ 0 & 0 & -1-j20 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0.4330 \\ 0.4176 + j0.5568 \\ 0.4176 - j0.5568 \end{bmatrix} u \\ y &= \begin{bmatrix} 1.1547 & -0.2155 + j0.2874 & -0.2155 - j0.2874 \end{bmatrix} \mathbf{z} \end{aligned}$$

where clearly the equivalent $\mathbf{A}, \mathbf{B}, \mathbf{C}$ matrices are complex.

It is often desirable to analyze a system using purely real arithmetic, and we now look at a method that transforms a system with complex conjugate eigenvalues to an "almost" diagonal form that has purely real elements in the $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ matrices.

Consider a diagonalized system, described by Eqs. (1), with a single complex conjugate eigenvalue pair, so that the $\mathbf{\Lambda}$ matrix is as shown in Eq. (13). Apply a further transformation $\mathbf{T}$ using the matrix

$$\mathbf{T} = \begin{bmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 0.5 & -j0.5 & & \\ & & 0.5 & j0.5 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{T}^{-1} = \begin{bmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & 1 & & \\ & & j & -j & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{bmatrix} \tag{14}$$

where $\mathbf{T}$ is an identity matrix in which the 2×2 block has replaced the unity terms on the diagonal on the rows corresponding to the complex eigenvalues. The resulting system is

$$\begin{aligned} \dot{\hat{\mathbf{z}}} &= \mathbf{T}^{-1}\mathbf{\Lambda}\mathbf{T}\hat{\mathbf{z}} + \mathbf{T}^{-1}\mathbf{M}^{-1}\mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{M}\mathbf{T}\hat{\mathbf{z}} + \mathbf{D}\mathbf{u} \end{aligned} \tag{15}$$

The resulting $\mathbf{A}$ matrix is

$$\hat{\mathbf{\Lambda}} = \mathbf{T}^{-1}\mathbf{\Lambda}\mathbf{T} = \begin{bmatrix} \lambda_1 & & & & & 0 \\ & \ddots & & & & \\ & & \sigma & \omega & & \\ & & -\omega & \sigma & & \\ & & & & \ddots & \\ 0 & & & & & \lambda_n \end{bmatrix} \tag{16}$$

where the real eigenvalues appear on the leading diagonal, but the complex eigenvalues appear as a 2×2 block consisting of the real and imaginary parts.

■ Example

Transform the diagonalized form of the previous example to have purely real matrices.

Solution: From Eq. (14) the transformation matrix $\mathbf{T}$ is

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & -j0.5 \\ 0 & 0.5 & j0.5 \end{bmatrix} \qquad \text{and} \qquad \mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & j & -j \end{bmatrix}$$

and the resulting system is

$$\begin{aligned} \dot{\hat{\mathbf{z}}} &= \mathbf{T}^{-1}\mathbf{\Lambda}\mathbf{T}\hat{\mathbf{z}} + \mathbf{T}^{-1}\mathbf{M}^{-1}\mathbf{B}\mathbf{u} \\ &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & -2 & -1 \end{bmatrix} \hat{\mathbf{z}} + \begin{bmatrix} 0.4330 \\ 0.8352 \\ -1.1136 \end{bmatrix} u \\ \mathbf{y} &= \mathbf{C}\mathbf{M}\mathbf{T}\hat{\mathbf{z}} + \mathbf{D}\mathbf{u} \\ &= \begin{bmatrix} 1.1547 & -0.2155 & 0.2874 \end{bmatrix} \hat{\mathbf{z}} + [\mathbf{0}]\, u \end{aligned}$$

where the system matrices are purely real.

The two transformations may be combined into a single transformation

$$\mathbf{P} = \mathbf{M}\mathbf{T}$$

so that the resulting system is

$$\begin{aligned} \dot{\hat{\mathbf{z}}} &= \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\hat{\mathbf{z}} + \mathbf{P}^{-1}\mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{P}\hat{\mathbf{z}} + \mathbf{D}\mathbf{u} \end{aligned} \tag{17}$$

where $\mathbf{x} = \mathbf{P}\hat{\mathbf{z}}$. Then from the definitions of $\mathbf{T}$ and $\mathbf{M}$ the transformation matrix $\mathbf{P}$ is

$$\mathbf{P} = \mathbf{M}\mathbf{T} = [v_1 \mid v_2 \mid \ldots \mid \Re\{v_i\} \mid \Im\{v_i\} \mid \ldots \mid v_n] \tag{18}$$

which is simply a modified form of the modal matrix $\mathbf{M}$ in which columns corresponding to any complex eigenvector pair has been replaced by a pair of columns containing the real and imaginary parts of those eigenvectors.

■ Example

Transform the system described in the first example to an almost diagonal system with real matrices in one step

Solution: From the first example

$$\mathbf{M} = \begin{bmatrix} 0.5774 & -0.1078 + j0.1437 & -0.1078 - j0.1437 \\ -0.5774 & -0.1796 - j0.3592 & -0.1796 + j0.3592 \\ 0.5774 & 0.8980 & 0.8980 \end{bmatrix}$$

The transformation matrix is found by replacing the second and third columns with the real and imaginary parts of the second column:

$$\mathbf{P} = \begin{bmatrix} 0.5774 & -0.1078 & 0.1437 \\ -0.5774 & -0.1796 & -0.3592 \\ 0.5774 & 0.8980 & 0 \end{bmatrix}$$

Then

$$\begin{aligned} \dot{\hat{\mathbf{z}}} &= \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\hat{\mathbf{z}} + \mathbf{P}^{-1}\mathbf{B}\mathbf{u} \\ &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & -2 & -1 \end{bmatrix}\hat{\mathbf{z}} + \begin{bmatrix} 0.4330 \\ 0.8352 \\ -1.1136 \end{bmatrix} u \\ \mathbf{y} &= \mathbf{C}\mathbf{P}\hat{\mathbf{z}} + \mathbf{D}\mathbf{u} \\ &= \begin{bmatrix} 1.1547 & -0.2155 & 0.2874 \end{bmatrix}\hat{\mathbf{z}} + [\mathbf{0}]\, u \end{aligned}$$

which is the same result as before, but found in a simple single transformation.

A system having complex conjugate eigenvalues can be transformed to an almost diagonal form with purely real elements in the system matrices using a transformation matrix derived from the modal matrix, but in which adjacent columns containing complex conjugate eigenvectors have been replaced by columns containing the real and imaginary parts.
The resulting equivalent A matrix will contain real eigenvalues on the diagonal, but will replace rows corresponding to complex conjugate eigenvalues with a 2×2 block containing the real and imaginary parts of the eigenvalue pair.