EE 561: Digital Control Systems
Problem Set # 1
Spring 2017

Due date: Feb 17, 2017

Question # 1 **(10 marks)**

a) Recall the following definition of the bilateral z-transform

$$\mathcal{Z}\left\{E(z)\right\} = \sum_{k=-\infty}^{\infty} e_k z^{-k}$$

assume that the signal e_k is given by the following geometric progression $e_k = r^k$ where $|r| < 1$. Does the bilateral z-transform converge for any region in the z-plane? Explain.

b) If your answer for part a) is no, can you propose a change in the signal $e_k = r^k$ for which the z-transform will converge? (If your answer to part a) is yes and you explained it correctly, you can skip this part for full credit).

c) In class we derived the following relation (while implicitly assuming the multiplication of the unit step)

$$\mathcal{Z}\left\{a^{k-1}\mathbb{1}(k-1)\right\} = \frac{1}{z-a}, \quad \text{where } \mathbb{1}(k) \text{ is the unit step.}$$

in light of the above expression, what is wrong with the following proof?

$$\mathcal{Z}\left\{a^{k-1}\mathbb{1}(k-1)\right\} = \sum_{k=0}^{\infty} a^{k-1} z^{-k}$$

$$= a^{-1} \sum_{k=0}^{\infty} a^k z^{-k} = a^{-1} \times \frac{1}{1-az^{-1}} = \frac{z}{a(z-a)}$$

after identifying the mistake, write down the altered proof.

d) what is the correct inverse transform of $\dfrac{z}{a(z-a)}$? (assume the bilateral transform)

Question # 2 **(20 marks)**

Consider the fibonacci sequence, which is given as the solution of the following recurrence

$$u_k = u_{k-1} + u_{k-2}; \quad u_0 = u_1 = 1;$$

your textbook solves the above equation through the undetermined coefficients method.

a) Obtain the values of u_2, u_3, and u_4, by posing the question as a linear algebra problem. Why do you not need to employ any finite difference approximations in the process?

b) Solve the recurrence using the Z-transform method. Use the unilateral transform.

c) Why is it not appropriate to solve the recurrence using the bilateral transform? If you were to indeed solve it using the bilateral transform, which recurrence would the solution actually correspond to?

Question # 3 **(10 marks)**

Recall the lecture in which we discussed the correspondence between the forward (backward) rectangular rule and the forward (backward) difference approximation.

a) Reformulate the centered difference approximation into its corresponding rectangular rule. Is it the trapezoidal rectangular rule?

b) Draw a pictorial representation of the rectangular rule obtained above.

Question # 4 **(10 marks)**

In class we analyzed the order of the error for the finite difference approximations of the first derivatives. For a sampling interval of T, the order of error for the forward and backward difference approximation is $\mathcal{O}(T)$, while the centered difference approximation has an error of $\mathcal{O}(T^2)$.
Using the same technique, obtain the order of error for the following approximation of the second derivative

$$\ddot{x}(t) \rightarrow \frac{x(t-T) - 2x(t) + x(t+T)}{T^2}$$

Question # 5 **(30 marks)**

Consider the following differential equation

$$\frac{d^2x}{dt^2} = -x; \quad x(0) = 1; \quad \frac{dx}{dt} = 0;$$

a) Find the complete solution $x(t)$ for the above equation.

b) Using the finite difference matrices, find x_1, x_2, x_3 and x_4. Assume a sampling period of $T = 0.1$. Approximate $\dot{x}$ as $\nabla_C x$ and $\ddot{x}$ as $\nabla_F \nabla_B x$. How accurate is the solution? Plot the error of the discrete solution with respect to the true solution obtained in part(a).

c) Under the same assumptions, use the Z-transform method to obtain x_1, x_2, x_3 and x_4. Obtain these values by numerically inverting the z-transform through long division.

d) Using MATLAB, plot the solutions as T increases from $T = 0.1$ in increments of 0.2. At what sampling time is the stability of the solution compromised?

Question # 6 **(20 marks)**

Using any method of your choice, solve the following difference equations

a)

$$u_{k+2} - 0.6u_{k+1} - 0.16u_k = 5e_{k+2}$$

where $u_{-1} = 0, u_{-2} = 25/4$ and $e_k = 4^{-k}\mathbb{1}(k)$.

b)

$$u_{k+2} + 6u_{k+1} + 9u_k = 0$$

where $u_{-1} = -1/3, u_{-2} = -2/9$.

c)

$$u_k + 4u_{k-2} = 0$$

where $u_{-1} = -1/(2\sqrt{2}), u_{-2} = 1/(4\sqrt{2})$.

Question # 7 **(10 marks)**

Consider a system with the following transfer function

$$H(z) = \frac{z - 0.5}{(z + 0.5)(z - 1)}$$

a) Write down the difference equation expressing the output u_k of the system to the input e_k.

b) Using any method of your choice, find the response of the system to the input $e_k = 3^{-(k+1)}\mathbb{1}(k)$.