Connectivity Graphs as Models of Local Interactions

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***Abstract*— In this paper, we study graphs that arise from certain sensory and communication limitations on the local interactions in multi-agent systems. In particular, we show that the set of graphs that can represent formations corresponds to a proper subset of all graphs and we denote such graphs as connectivity graphs. Such graphs have a special structure that allows them to be composed from a small number of atomic crossing generators using a certain kind of graph amalgamation. This structure allows us to give connectivity graphs a useful topological characterization in terms of their simplicial complexes.**

I. Introduction

The problem of coordinating multiple mobile robots is one in which a £nite representation of the con£guration space appears naturally, namely by using graph-theoretic models for describing the local interactions in the formation. In other words, these models serve as a bridge between the continuous and the discrete when trying to manage the design-complexity associated with formation control. Notable results along these lines have been presented in [1], [2], [3], [4], [5]. The conclusion to be drawn from these research efforts is that a number of questions can be answered in a natural way by abstracting away the continuous dynamics of the individual agents. In particular, if the existence of an edge between two agents corresponds to certain geometric relations such as spatial closeness, then the graphs are really dynamic in the sense that the time-driven evolution of the con£guration space may result in the production of a new graph. We show here that the graphs that can represent formations do in fact correspond to a proper subset of all graphs, denoted by the set of connectivity graphs (Section II), and it is possible to give a topological characterization of such graphs in terms of their geometrical structure (Section III) and their simplicial complexes (Section IV), followed by some conclusions (Section V).

II. Formations and Connectivity Graphs

Graphs can model local interactions between agents, when individual agents are constrained by limited knowledge of other agents. Here, we consider formations in which the primary limitation of perception for each agent is the range of its sensor. Suppose we have N such agents with identical dynamics evolving on $\mathbb{R}^2$. Each agent is equipped with an identical range limited sensor by which it can sense the position of other agents within a distance δ. Let the position of each agent be $x_n \in \mathbb{R}^2$. The con£guration space $\mathcal{C}^N(\mathbb{R}^2)$ of the agent formation is made up of all ordered N-tuples in $\mathbb{R}^2$, with the property that no two points coincide, i.e. $\mathcal{C}^N(\mathbb{R}^2) = (\mathbb{R}^2 \times \mathbb{R}^2 \times \dots \mathbb{R}^2) - \Delta$, where $\Delta = \{(x_1, x_2, \dots, x_N) : x_i = x_j \text{ for some } i \neq j\}$. The evolution of the formation can be represented as a trajectory $\mathcal{F} : \mathbb{R}_+ \to \mathcal{C}^N(\mathbb{R}^2)$, usually written as $\mathcal{F}(t) = (x_1(t), x_2(t), \dots x_N(t))$ to signify time evolution. The spatial relationship between agents can be represented as a graph in which the vertices of the graph represent the agents, and the pair of vertices on each edge tells us that the corresponding agents are within sensor range δ of each other. However several formations may give the same graph. We make these ideas precise as follows.

De£nition 2.1 (Connectivity Graph of a Formation): Let $\mathcal{G}_N$ denote the space of all possible graphs that can be formed on N vertices $V = \{v_1, v_2, \dots, v_N\}$. Then we can de£ne a function $\Phi_N : \mathcal{C}^N(\mathbb{R}^2) \to \mathcal{G}_N$, with $\Phi_N(\mathcal{F}(t)) = \mathcal{G}(t)$, where $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t)) \in \mathcal{G}_N$ is the *connectivity graph* of the formation $\mathcal{F}(t)$. $v_i \in \mathcal{V}$ represents agent i at position x_i, and $\mathcal{E}(t)$ denotes the edges of the graph. $e_{ij}(t) = e_{ji}(t) \in \mathcal{E}(t)$ if and only if $\|x_i(t) - x_j(t)\| \leq \delta \; i \neq j$.

One can observe that these graphs are *simple* by construction i.e. there are no loops or parallel edges. They are always undirected because the sensor ranges are identical. Furthermore, the motion of agents in a formation may result in the removal or addition of edges in the graph, which makes $\mathcal{G}(t)$ a dynamic structure. Most importantly, every graph in $\mathcal{G}_N$ is not a connectivity graph. This last observation is not as obvious as the others and it is analyzed in detail below. While many researchers have referred to graphs of formations as their models; [1], [4]; the issue of whether an arbitrary graph corresponds to a some agent formation has been mostly overlooked. A *realization* of a graph $\mathcal{G} \in \mathcal{G}_N$ is a formation $\mathcal{F} \in \mathcal{C}^N(\mathbb{R}^2)$, such that $\Phi_N(\mathcal{F}) = \mathcal{G}$. An arbitrary graph $\mathcal{G} \in \mathcal{G}_N$ can therefore

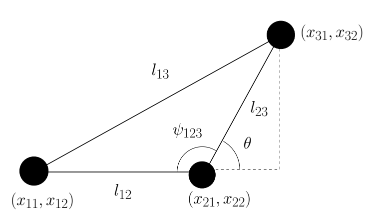


Fig. 1. Depicted are three robots and their inter-robot distances.

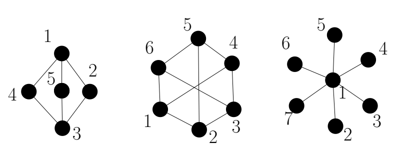
be *realized* as a connectivity graph in $\mathcal{C}^N(\mathbb{R}^2)$ if $\Phi_N^{-1}(\mathcal{G})$ is nonempty. We denote by $\mathcal{G}_{N,\delta} \subseteq \mathcal{G}_N$, the space of all possible graphs on N agents with sensor range δ, that can be realized in $\mathcal{C}^N(\mathbb{R}^2)$. Let us start by analyzing this space for small values of N. The cases for $N = 1$ and $N = 2$ are trivial. Similarly, for $N = 3$, several trivial constructions can be obtained for various connected and disconnected graphs on 3 vertices. Consider the situation in Figure 1, where the 3 agents are positioned at the points marked by circles. Let each position x_i be given by its Cartesian coordinate pair $(x_{i1}, x_{i2})^T$. For notational convenience let $\|x_1 - x_2\| = l_{12}$, $\|x_2 - x_3\| = l_{23}$ and $|x_1 - x_3\| = l_{13}$. Also let θ and ψ_{123} the angles between various line segments shown in the £gure. In general, any connectivity graph on N vertices gives various constraints on the relative positions of individual agents in the con£guration space $\mathcal{C}^N(\mathbb{R}^2)$. We explain below that in the case of a connected graph on 3 vertices, the constraints on positions x_1, x_2 and x_3 correspond to a single constraint on the angle ψ_{123}, when the the agents are positioned as shown in the £gure. This simple observation will subsequently lead to some interesting properties of the connectivity graphs and their realizations. Suppose we are considering the line graph on 3 vertices in Figure 3, then the given geometrical con£guration corresponds to this graph if $l_{12} \leq \delta, l_{23} \leq \delta$, and $l_{13} > \delta$. Moreover we can write

$$\begin{aligned} l_{13}^2 &= (l_{12} + l_{23}\cos\theta)^2 + (l_{23}\sin\theta)^2, \\ &= l_{12}^2 + l_{23}^2 + 2l_{23}l_{12}\cos\theta. \end{aligned}$$

If $l_{13} > \delta$ then $\cos\theta > \frac{\delta^2 - l_{12}^2 - l_{23}^2}{2l_{23}l_{12}}$. It is easy to see that the term on the right has a minimum corresponding to the maximum values of $l_{12} = l_{23} = \delta$. This implies that $\cos\theta > -\frac{1}{2}$, which means that $\theta \in [-\frac{2\pi}{3}, \frac{2\pi}{3}]$. So the smaller angle between l_{12} and l_{23}, $\psi_{123} = \pi - \theta > \frac{\pi}{3}$, for all $0 < l_{12}, l_{23} \leq \delta$ and $l_{13} > \delta$. Therefore, whenever we have two edges e_{ij} and e_{ik} in a connectivity graph that share a vertex v_i in such a way that there is no edge between vertices v_j and v_k, then

$$\psi_{j,i,k} \triangleq \cos^{-1}\left(\frac{< x_j - x_i, x_i - x_k >}{\|x_j - x_i\|\|x_i - x_k\|}\right) > \frac{\pi}{3} \qquad (1)$$

Denote by S_N the "star graph" in $\mathcal{G}_N$ i.e. the graph which has $N-1$ vertices $v_2, v_3 \ldots v_N$ of degree 1 and one vertex v_1 with degree $N-1$. An example of such a graph is shown in Figure 2.a. Also denote by $\diamondsuit_5$ and $\diamondsuit_6$, the graphs in $\mathcal{G}_5$ and $\mathcal{G}_6$ respectively, as drawn in Figures 2.a and 2.b.



(a). (b). (c).

Fig. 2. Graphs $\diamondsuit_5$, $\diamondsuit_6$ and S_7, that are not connectivity graphs

Proposition 2.1: The graphs $\diamondsuit_5 \in \mathcal{G}_5$ and $\diamondsuit_6 \in \mathcal{G}_6$ do not belong to $\mathcal{G}_{5,\delta}$ and $\mathcal{G}_{6,\delta}$ respectively.

Proof: Suppose that to the contrary $\diamondsuit_5 \in \mathcal{G}_{5,\delta}$ then there exists some realization $\mathcal{F} = (x_1, x_2, \cdots x_5) \in \mathcal{C}^5(\mathbb{R}^2)$ such that $\Phi_5(\mathcal{F}) = \diamondsuit_5$. From Equation 1 it follows that the angles ψ_{415}, ψ_{512}, ψ_{123}, ψ_{235}, ψ_{534} and ψ_{341} are all greater than $\frac{\pi}{3}$. Therefore, $\psi_{415} + \psi_{512} + \psi_{123} + \psi_{235} + \psi_{534} + \psi_{341} > 6\left(\frac{\pi}{3}\right) = 2\pi$. But since $x_1, x_2, x_3, x_4 \in \mathbb{R}^2$ are vertices of a polygon of 4 sides, $\psi_{415} + \psi_{512} + \psi_{123} + \psi_{235} + \psi_{534} + \psi_{341} = 2\pi$, which is a contradiction. Therefore $\diamondsuit_5 \notin \mathcal{G}_{5,\delta}$. The proof for $\diamondsuit_6 \notin \mathcal{G}_{6,\delta}$ is very similar and can be found in [12]. ∎

Proposition 2.2: $\mathcal{S}_N \in \mathcal{G}_N$ does not belong to $\mathcal{G}_{N,\delta}$ for $N > 6$.

Proof: Suppose to the contrary, $\mathcal{S}_N \in \mathcal{G}_{N,\delta}$. If $deg(v_1) = N - 1$, and $(x_1, \cdots x_N) \in \mathcal{C}^N(\mathbb{R}^2)$ is a realization then $\psi_{i,1,j} > \frac{\pi}{3}$ for all $2 \leq i, j \leq N$. We have, $\psi_{2,1,3} + \psi_{3,1,4} + \cdots + \psi_{N-2,1,N-1} + \psi_{N-1,1,N} + \psi_{N,1,2} > (N-1)\pi/3$. If $N > 6$ then this sum is strictly greater than 2π. However by the given setup, this sum should be exactly equal to 2π. Therefore, by this contradiction $\mathcal{S}_N \notin \mathcal{G}_{N,\delta}$ for $N > 6$. ∎

There are of course many other examples of realizable and non-realizable connectivity graphs. If a graph is completely disconnected, it means that the distance between any two agents in the formation is separated by more than δ. This can easily be achieved by placing the vertices one by one in such a way that x_i does not belong to $\bigcup_{j=1}^{i-1} \mathcal{B}_\delta(x_j)$, where $\mathcal{B}_\delta(x)$ is the closed ball of radius δ centered at x. This observation can be further generalized as follows.

Lemma 2.1: A graph $\mathcal{G} \in \mathcal{G}_{N,\delta}$ if and only if each of its connected component $\mathcal{G}_i \in \mathcal{G}_{M_i}$ is realizable in some $\mathcal{G}_{M_i,\delta}$, $M_i < N$.

We omit a formal proof here for brevity but the concept is easy to understand. We saw earlier that completely disconnected graphs are trivially realizable by placing the agents further than δ from one another. If $\mathcal{G} \in \mathcal{G}_N$ has many disjoint connected components, say $\{\mathcal{G}_i\}$, we can place each connected component 'far away' from all other components so that none of the agents in one component can sense agents in other connected component. By this construction we have a realization for G if and only if all G_i have realizations in their respective spaces $G_{M_i,\delta}$.

Theorem 2.1: $\mathcal{G}_{N,\delta}$ is a proper subspace of $\mathcal{G}_N$ if and only if $N \geq 5$.

Proof: In order to prove that $\mathcal{G}_{N,\delta}$ is a proper subspace of $\mathcal{G}_N$ for some N, it is enough to show that $\Phi : C^N(\mathbb{R}^2) \to \mathcal{G}_N$ is not onto. Therefore we need to provide a graph $\mathcal{G} \in \mathcal{G}_N$ such that $\Phi^{-1}(\mathcal{G}) = \emptyset$. From Proposition 2.1, we have examples of graphs that are not realizable in $\mathcal{G}_{5,\delta}$ and $\mathcal{G}_{6,\delta}$. For $N \geq 7$ the star graphs $\mathcal{S}_N$ of Proposition 2.2 provide the examples of graphs that cannot be realized as connectivity graphs in $\mathcal{G}_{N,\delta}$. This proves that $\mathcal{G}_{N,\delta}$ is a proper subspace of $\mathcal{G}_N$ if $N \geq 5$.

To prove that every graph in $\mathcal{G}_N$, for $N < 5$, is realizable in $\mathcal{G}_{N,\delta}$, we have to enumerate all possible graphs for $N < 5$

and give realizations for each graph. Since we are dealing with a small number $N(< 5)$, the enumeration strategy works well. The number of possible graphs to check can be further reduced by noting that we need to consider only connected graphs. The justi£cation for this comes from Lemma 2.1 given above. In fact, from [6] we know what these graphs are, and they together with their realizations are given in Figures 3 and 4, which completes the proof. ■

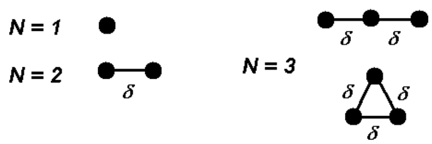


Fig. 3. Possible realizations for all $G \in \mathcal{G}_{N,\delta}$ for $N \leq 3$

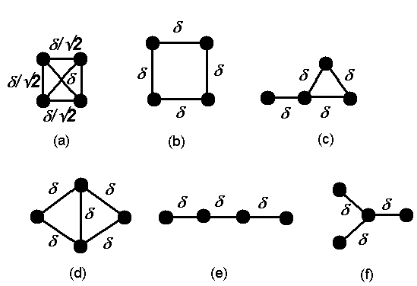


Fig. 4. Possible realizations for all connectivity graphs in $\mathcal{G}_{4,\delta}$.

Corollary 2.1: If each connected component $\mathcal{G}_i$ of a graph $\mathcal{G} \in \mathcal{G}_N$ belongs to $\mathcal{G}_{M_i}$, $M_i < 5$ then the graph has a realization in $\mathcal{G}_{N,\delta}$.

Formations can produce a wide variety of graphs including those that have disconnected components. However the problem of switching between different formations or of £nding interesting structures within a formation of sensor range limited agents can only be tackled if no sub-formation of agents is totally isolated from the rest of the formation [5]. This means that the connectivity graph $\mathcal{G}(t)$ of the formation $\mathcal{F}(t)$ should always remain *connected* (in the sense of connected graphs) for all time t.

III. Geometric Structure of Connectivity Graphs

In this section we produce some results about the geometric structure of connectivity graphs. We will see that the graphs are made up of atomic graphs which when combined in a certain way produce more complex graphs. If a given formation $\mathcal{F} = (x_1, x_2, \ldots x_N) \in \mathcal{C}^N(\mathbb{R}^2)$ has the connectivity graph $G = (\mathcal{V}, \mathcal{E}) = \Phi_N(\mathcal{F}(t))$, then each edge $e_k = \{v_{k_1}, v_{k_2}\} \in \mathcal{E}$ can be mapped to $\mathbb{R}^2$ by a map $f_k : \mathbb{R} \to \mathbb{R}^2$ given by $f_k(s) = sx_{k_1} + (1-s)x_{k_2}$ for $s \in [0, 1]$. We call the image of the mapping f_k, the *image* of the edge in $\mathbb{R}^2$. The *image of a formation*, $I_{\mathcal{F}} \in \mathbb{R}^2$ can

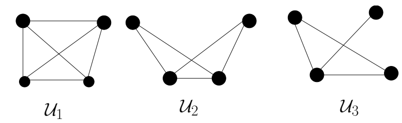


Fig. 5. Crossing Generators.

be de£ned as the union of the images of all edges in the connectivity graph of the formation.

$$I_{\mathcal{F}} = \bigcup_{e_k \in \mathcal{E}} f_k([0, 1]) \subset \mathbb{R}^2. \tag{2}$$

Note that the image is constructed by mapping each vertex v_i of the graph to its position x_i and each edge $e_k = \{v_{k_1}, v_{k_2}\}$ to a line segment $sx_{k_1} + (1-s)x_{k_2}$, for $s \in [0, 1]$, in $\mathbb{R}^2$. If it is clear from the context, what formation is under consideration, we will often write $I_{\mathcal{F}}$ as I_G, where $G = \Phi(\mathcal{F})$. Sometimes it will be convenient to describe the image of a subgraph $H = (\mathcal{E}_H, \mathcal{V}_H)$ of the connectivity graph G of formation $\mathcal{F}$, where $\mathcal{E}_H \subset \mathcal{E}$ and $\mathcal{V}_H \subset \mathcal{V}$. In this case, we refer to the image of the subgraph H as

$$I_H = \bigcup_{e_k \in \mathcal{E}_H} f_k([0, 1]) \subset \mathbb{R}^2, \; H \subseteq G = \Phi_N(\mathcal{F}) \in \mathcal{G}_{N,\delta}. \tag{3}$$

The image is thus what a graph would "look like" if drawn in the plane. Note that this is different from the concept of planar graphs [7] or imbedding graphs in $\mathbb{R}^2$ where edges are not necessarily mapped to straight lines. Two edges $e_i, e_j \in \mathcal{E}$ of a graph are said to be *crossing* if $f_i(s) = f_j(t)$ for some $s, t \in (0, 1)$ and the set $f_i([0, 1]) \cap f_j([0, 1])$ has dimension 0. According to this de£nition, edge intersection at some vertex of the two edges does not count as a crossing. The condition that the intersection set is of dimension 0, rules out edge intersections of collinear points. For convenience denote by $e_i \ltimes e_j = \text{true}$, if $e_i, e_k \in \mathcal{E}$ are crossing edges. It should moreover be noted that the points in the image I_G can be categorized as smooth or non-smooth, where smoothness is de£ned in the setting of smooth manifolds. Any point x in the image that is not one of the robot positions $\{x_i\}_{i=1}^N$ or the crossing points is smooth.

Proposition 3.1: An image of a formation of 4 vertices has a pair of crossing edges only if its connectivity graph is isomorphic to either $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ as shown in Figure 5.

Proof: From the Figures in 4, we see that only those connectivity graphs that are isomorphic to the graphs in Fig 5, namely those in 4.a 4.c and 4.d can be realized with crossing edges. For all other graphs in Figure 4, any attempt to create an image with crossing edges results in a violation of the constraints that de£ne these graphs. ■

The three graphs $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ are called the *crossing generators* of all connectivity graphs. If G is a connectivity graph of a formation $\mathcal{F}$, and has two edges e_1, e_2 such that $e_1 \ltimes e_2 = \text{true}$, then there always exists a subgraph of G that e_1 and e_2 and is isomorphic to one of the crossing generators $\mathcal{U}_1$,$\mathcal{U}_2$ or $\mathcal{U}_3$. We now de£ne the following operation.

De£nition 3.1 (Δ-Amalgamation of crossing generators): If $\mathcal{U}_i$,$\mathcal{U}_j \in \mathcal{G}_4$ are two crossing generators, $H \subset \mathcal{U}_i$ and $H' \subset \mathcal{U}_j$ are subgraphs s.t. $H, H' \simeq \mathcal{K}_3$ (the complete graph on 3 vertices), and there is an isomorphism $\Delta : H \rightarrow H'$ between the respective subgraphs, then their amalgamation (as standard amalgamation of two graphs [7]) according to the isomorphism Δ is called a Δ-amalgamation, denoted by $\mathcal{U}_i *_\Delta \mathcal{U}_j$.

In the context of connectivity graphs of formations, Δ - amalgamation is used as a description for unions of the type, $\bigcup_i I_{G_i}$, where each $G_i \subseteq G$, each G_i is a valid connectivity graph in $\mathcal{G}_{4,\delta}$ and each $G_i \simeq \mathcal{U}_j$ for some $1 \leq j \leq 3$. The Δ-amalgamation $G_{i_1} *_\Delta G_{i_2}$ is well de£ned for connectivity graphs, if $G_{i_1} *_\Delta G_{i_2}$ is a valid connectivity graph in $\mathcal{G}^c_{5,\delta}$. Generalizing this for an arbitrary number of amalgamations, the operation is well de£ned if

$$\underbrace{G_{i_1} *_\Delta G_{i_2} *_\Delta \ldots G_{i_k}}_{(k-1)-amalgamations} \in \mathcal{G}^c_{4+k,\delta} \tag{4}$$

It should be mentioned that there are several Δ- amalgamations between any two crossing generators, depending on the choice of H and H'. The Δ-amalgamation operation for arbitrary graphs can be repeated to generate a whole family of graphs from the crossing generators. If we let $\Sigma = \sigma_1.\sigma_2.\ldots.\sigma_K$ be a £nite string de£ned over $\{1, 2, 3\}$, then we denote a member of this family as:

$$\mathcal{G}_\Sigma = \mathcal{U}_{\sigma_1} *_\Delta \mathcal{U}_{\sigma_2} *_\Delta \ldots *_\Delta \mathcal{U}_{\sigma_K} \tag{5}$$

If we have repeated Δ-amalgamations of subgraphs of a connectivity graph, as in (4), there always exists a £nite string Σ such that

$$\mathcal{G}_\Sigma \simeq G_{i_1} *_\Delta G_{i_2} *_\Delta \ldots G_{i_k} \in \mathcal{G}^c_{4+k,\delta} \tag{6}$$

Each well de£ned repeated Δ-amalgamation, as de£ned in (6), is called an *Atomic Crossing Graph*. Let $I_{\mathcal{G}_\Sigma}$ denote the image of a atomic crossing graph by referring to its isomorphic graph $\mathcal{G}_\Sigma$, when the details of the Δ-amalgamations is clear from context. Given a formation $\mathcal{F}$ and its connectivity graph $\Phi(\mathcal{F}) = G = (\mathcal{E}, \mathcal{V})$, let $\mathcal{E}_\times \subseteq \mathcal{E}$ be the set of all crossing edges, and $\mathcal{V} \supseteq \mathcal{V}_\times = \{v \in \mathcal{V} \mid v \in e$ for some $e \in \mathcal{E}_\times\}$, then $H_\times = (\mathcal{E}_\times, \mathcal{V}_\times)$ is the subgraph of G made up of all crossing edges of the connectivity ‘graph. We denote by $H_\triangle = (\mathcal{E} \setminus \mathcal{E}_\times, \mathcal{V})$, the subgraph of G consisting of all non-crossing edges so that $G = H_\times \cup H_\triangle$.

Proposition 3.2: There exist a set of atomic crossing graphs $\{\mathcal{G}_{\Sigma_j}\}_{j \in J}$ such that

$$I_{H_\times} \subset \bigcup_{j \in J} I_{\mathcal{G}_{\Sigma_j}} \subseteq I_G \subset \mathbb{R}^2, \tag{7}$$

where J is some £nite indexing set, and each $x \in \left(I_{\mathcal{G}_{\Sigma_i}} \cap I_{\mathcal{G}_{\Sigma_j}}\right) \setminus \{x_k\}_{k=1}^N$ for $i, j \in J$, is smooth.

Proof: With slight abuse of notation, let $e \in G = (\mathcal{E}, \mathcal{V})$ denote that $e \in \mathcal{E}$. If $e_i \ltimes e_j =$ true, then denote by $H_q(e_i, e_j)$ the subgraph of G such that $H_q(e_i, e_j) \simeq \mathcal{U}_q$ for some $1 \leq q \leq 3$ and $e_i, e_j \in H_q(e_i, e_j)$. Also denote by $\mathcal{E}_{\Sigma_j}$ the set of edges for the graph $\mathcal{G}_{\Sigma_j}$. We now give the following algorithm to provide a constructive way of obtaining the atomic crossing graphs.

Algorithm 3.1: .

- A $j \leftarrow 0$
- B $\mathcal{E}_\times = \{e \in \mathcal{E} \mid e \ltimes e' = \text{true for some } e' \in \mathcal{E}\}$
- C while $\mathcal{E}_\times \neq \emptyset$
  1) $j \leftarrow j + 1$
  2) $k \leftarrow 1$
  3) Pick $e_m \in \mathcal{E}_\times$
  4) Pick $e_p \in \mathcal{E}_\times$ such that $e_m \ltimes e_p =$ true
  5) $\mathcal{E}_\times \leftarrow \mathcal{E}_\times \setminus \{e_p, e_m\}$
  6) $\sigma_k = \arg\min_{1 \leq q \leq 3} (H_q(e_m, e_p))$
  7) $\Sigma_j \leftarrow \sigma_k$
  8) $\mathcal{G}_{\Sigma_j} \leftarrow H_{\sigma_k}(e_m, e_p)$
  9) $\mathcal{E}_{j_\circ} = \{e \in \mathcal{E}_\times \mid e \in H_{\sigma_k}(e_m, e_p)\}$
  10) $\mathcal{E}_\times \leftarrow \mathcal{E}_\times \setminus \mathcal{E}_{j_\circ}$
  11) $\mathcal{E}_{j_\times} = \{e \in \mathcal{E}_\times \mid e \ltimes e_l = \text{true for some } e_l \in \mathcal{E}_{\Sigma_j}\}$
  12) while $\mathcal{E}_{j_\times} \neq \emptyset$
      a) Pick $e_r \in \mathcal{E}_{j_\times}$
      b) Pick $e_s \in \mathcal{E}_{\Sigma_j}$ such that $e_r \ltimes e_s =$ true
      c) $\mathcal{E}_\times \leftarrow \mathcal{E}_\times \setminus \{e_r, e_s\}$
      d) $\sigma_{k+1} = \arg\min_{1 \leq q \leq 3} (H_q(e_r, e_s))$
      e) $\Sigma_j \leftarrow \Sigma_j \sigma_{k+1} = \sigma_1 \sigma_2 \ldots \sigma_k \sigma_{k+1}$
      f) $\mathcal{G}_{\Sigma_j} \leftarrow \mathcal{G}_{\Sigma_j} *_\Delta H_{\sigma_{k+1}}(e_r, e_s)$
      g) $\mathcal{E}_{j_\circ} = \{e \in \mathcal{E}_\times \mid e \in \mathcal{E}_{j_\times} \wedge e \in H_{\sigma_{k+1}}(e_r, e_s)\}$
      h) $\mathcal{E}_\times \leftarrow \mathcal{E}_\times \setminus \mathcal{E}_{j_\circ}$
      i) $\mathcal{E}_{j_\times} = \{e \in \mathcal{E}_\times \mid e \ltimes e_l = \text{true for some } e_l \in \mathcal{E}_{\Sigma_j}\}$
      j) $k \leftarrow k + 1$
  13) end
- D end

A detailed working of the algorithm has been explained in [12]. The main point to note is that since there are only £nitely many crossing edges, denoted by $\mathcal{E}_\times$, this algorithm always terminates in less than $|\mathcal{E}_\times|/2$ number of Δ-amalgamations. Therefore, all crossing edges are eventually absorbed into one of $\mathcal{G}_{\Sigma_i}$ in a £nite number of steps so that $H_\times \subset \cup_{j \in J} \mathcal{G}_{\Sigma_j}$, and hence $I_{H_\times} \subset \bigcup_{j \in J} I_{\mathcal{G}_{\Sigma_j}} \subseteq I_G \subset \mathbb{R}^2$. ∎

The above discussion gives a decomposition of connectivity graphs in terms of crossing and non-crossing edges. These properties will become useful for obtaining a simplicial representation of connectivity graphs, which will subsequently help in understanding the topological shape of formations as discussed in the following section.

IV. Topological Characterization of Connectivity Graphs

It is a well known fact from algebraic topology that the study of topological shapes of compact closed manifolds is synonymous to the study of triangulations of manifolds.

These triangulations are called simplicial complexes. We refer the reader to [8] and for an introduction to simplicial complexes and their importance in determining topology. A simplicial complex of dimension 1 can be thought of as a graph whose image has no crossing edges. The only non-smooth points in the complex are the images of the vertices. Therefore, if some appropriate crossing edges are removed from a connectivity graph, its image is a well-de£ned simplicial complex. Proposition 3.2 leads to some conclusions along these lines. All points in the image I_{H_Δ} are smooth except the vertex points. This makes I_{H_Δ} a well de£ned simplicial complex of dimension 1. Therefore the problem of obtaining a simplicial representation for the graph is reduced to £nding one for $I_{H_\times}$. If the image of each atomic crossing graph can be converted into a simplicial complex, by removal of images of crossing edges, then the union of the individual simplicial complexes would be a well-de£ned simplicial complex, as guaranteed by Proposition 3.2. Therefore the problem of obtaining a simplicial complex from a connectivity graph can be solved if each sub-problem of obtaining a simplicial complex for each atomic crossing graph can be solved. Every connectivity graph $G \in \mathcal{G}_{N,\delta}$ has at least one subgraph G_s which induces a well de£ned simplicial complex $K_{G_s} = (\mathcal{V}, \mathcal{S})$. We call G_s a *simplicial subgraph* of G. If a pair of vertices $S = \{v_i, v_j\}$ de£nes an edge in G_s then $S \in \mathcal{S}$. In other words every edge induces a 1-simplex in K_{G_s}. The subgraph of non-crossing edges $H_\Delta \subseteq G$, is also an example of a simplicial subgraph of G.

De£nition 4.1 (Maximal Simplicial Subgraph): A subgraph $G^* \subset G$ is said to be a maximal simplicial subgraph of G if there does not exist a simplicial subgraph of G that properly contains G^*.

Before developing a method to obtain this maximal simplicial subgraph, a few points should be noted. In order to preserve maximality, the removal of any non-crossing edge in the graph is not allowed. Care has to be taken during the removal of crossing edges, as the removal of one crossing edge may result in the removal of all non-smooth points on another crossing edge, making the later non-crossing. Therefore the order in which crossing edges are removed is important. The maximal simplicial subgraph of a connectivity graph is not unique and depends on the order in which crossing edges are removed. We begin by considering the problem of obtaining maximal simplicial subgraphs of atomic crossing graphs.

Proposition 4.1: There exists an algorithm to obtain a maximal simplicial subgraph of every atomic crossing graph.

Proof: Let an atomic crossing graph $G = (\mathcal{E}, \mathcal{V})$ be isomorphic to G_Σ, as given by 6 or obtained by executing Algorithm 3.1, so that the string $\Sigma = \sigma_1\sigma_2\cdots\sigma_K$ gives the order of Δ-amalgamations in the atomic graph,

$$G_\Sigma \simeq G = G_1 *_\Delta G_2 *_\Delta \ldots G_K \in \mathcal{G}^c_{4+K,\delta},$$

where $G_k \simeq \mathcal{U}_{\sigma_k}$ for $1 \le k \le K$. Now execute the following algorithm.

Algorithm 4.1: .

1) $\mathcal{E}^* \leftarrow \mathcal{E}$
2) $G^* \leftarrow (\mathcal{E}^*, \mathcal{V})$
3) for $k = K$ to 1
   a) $\mathcal{E}_\times = \{e \in G_k \mid e \ltimes e' = \text{true for some } e' \in G^* \setminus G_k\}$
   b) $\mathcal{E}^* \leftarrow \mathcal{E}^* \setminus \mathcal{E}_\times$
   c) $G^* \leftarrow (\mathcal{E}^*, \mathcal{V})$
   d) $\mathcal{E}_\circ = \{e \in G^* \cap G_k \mid e \ltimes e' = \text{true for some } e' \in G^* \cap G_k\}$
   e) if $|\mathcal{E}_\circ| = 2$ then
      i) $\{e_1, e_2\} \leftarrow \mathcal{E}_\circ$
      ii) $\mathcal{E}^* \leftarrow \mathcal{E}^* \setminus e_1$
      iii) $G^* \leftarrow (\mathcal{E}^*, \mathcal{V})$
   f) end
4) end

Again, we omit the detailed working of the algorithm, and refer to [12]. It is enough to note that by construction, the algorithm ful£lls all conditions for maximality as enumerated above. Therefore the graph $G^* = (\mathcal{E}^*, \mathcal{V})$ obtained at the end of the algorithm is indeed the maximal simplicial subgraph of G. ■

Theorem 4.1: A maximal simplicial subgraph G^* of a connectivity graph $G \in \mathcal{G}_{N,\delta}$ is given by the union,

$$G^* = \left(\bigcup_i G^*_{\Sigma_i}\right) \cup H_\Delta. \tag{8}$$

Proof: By Proposition 3.2 any crossing edge of a connectivity graph is contained in some atomic crossing graph G_{Σ_i} generated by repeated Δ-amalgamations. Also by the same result, the intersection of the images of any two atomic crossing graphs G_{Σ_i} and G_{Σ_j} is made up of only smooth points (except the vertices). This means that the intersection of the two graphs has only non-crossing edges in common and the removal of crossing edges in one atomic crossing graph does not effect the crossing edges in the other graph. If we now obtain the maximal simplicial subgraph $G^*_{\Sigma_i}$ of each atomic crossing graph G_{Σ_i}, it does not result in the removal of any non-crossing edge in $G_{\Sigma_i} \cap G_{\Sigma_j}$. Therefore, if we obtain the maximal simplicial subgraph of each atomic crossing graph by executing Algorithm 4.1, then their union $\cup_i G^*_{\Sigma_i}$ is also maximal. Finally, the subgraph H_Δ contains no crossing edges and is already a maximal simplicial subgraph, thereby making G^* in Equation 8 a maximal simplicial subgraph of G. ■

It should be noted that maximal simplicial subgraph of a connectivity graph need not be unique and depends on the order in which crossing edges are removed, i.e. the order in which Δ-amalgamations are repeated to obtain each atomic crossing graph. While the maximality condition captures the maximal simplicial structure in the connectivity graph, we can further see that any 3 vertices that span a triangle in G_s, can span a 2-simplex in the image. Therefore we have the following.

De£nition 4.2 (Maximal Simplicial Complex): $M_{G_*} = (\mathcal{V}, \mathcal{S})$ is the maximal simplicial complex spanned by a connectivity graph if:

- $\mathcal{S}_1 \subseteq \mathcal{S}$, where $(\mathcal{V}, \mathcal{S}_1)$ is the 1-simplicial complex of the maximal simplicial subgraph $G_* \subset G$.
- If a set of any three vertices $L = \{v_i, v_j, v_k\}$ form a cycle in $(\mathcal{V}, \mathcal{S}_1)$ then $L \in \mathcal{S}$

The set of vertices $S = \{v_i, v_j, v_k\}$, that form a cycle in G_* induces a 2-simplex in K. Therefore M_{G_*} is a 2-complex made up of:

1) 1-simplexes induced by the edges of the graph
2) 2-simplexes induced by the cycles of 3 vertices of the graph

It may happen that M_{G_*} has no 2-simplexes. In this case the 1-skeleton is the complex itself case either. M_{G^*} is in fact the object associated with the topological shape of the formation. Some comments are appropriate at this point to explain why we have associated the topological characterization of the formation with the maximal simplicial complex spanned by its connectivity graph. Compare the lower connectivity graph on 3 vertices in Figure 3 with the graph in Figure 4.b. Both have a cyclic ring-like structure, which apparently makes them topologically equivalent. However, there is a subtle difference between the two, if we also desire that this structure is a decentralized multi-agent system. In the former example, all nodes interact directly with each other, whereas in the latter any node interacts directly with its two adjacent nodes only. Therefore the absence of an edge between opposite nodes creates a "hole" in the topological shape. By the method described in De£nition 4.2, there would be a 2-simplex attached to the image of the graph of Figure 3 to get its maximal simplicial complex, making it a topological object of genus 0. On the other hand the maximal simplicial complex of the graph in Figure 4.b. would still have a hole (genus 1). De£nition 4.2 lets us expand this point of view to more complex graphs. Once we have the simplicial complex of a graph, it can be studied using tools from standard algebraic topology to obtain its genus, fundamental groups, homological groups, etc. It will be appropriate to mention, that the entire machinery presented here for decomposing connectivity graphs into simplicial complexes becomes irrelevant, if the computations are performed in a centralized manner. The tools developed above can be used to obtain certain global properties of formations using decentralized algorithms, suitable for implementation on scalable, totally decentralized multi-agent systems. The details of these applications can be found in [10], [11] and [12].

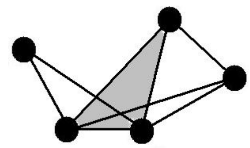


Fig. 6. Δ-amalgamation of $\mathcal{U}_1$ and $\mathcal{U}_2$.

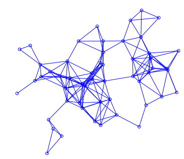


Fig. 7. A connectivity graph G.

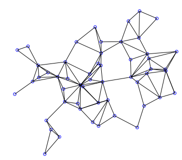


Fig. 8. The maximal simplicial subgraph G^*.

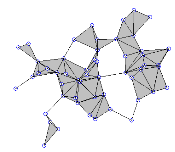


Fig. 9. The maximal simplicial complex M_{G^*} spanned by the connectivity graph of Figure 7.

V. CONCLUSIONS

The connectivity graphs of formations, that arise due to sensory or communication limitations of individual agents have a rich set of structural and topological properties. These graphs are an important abstraction, as they let us capture various structural properties of the formations without referring to the actual positions of the agents. As the number of agents in a formation is increased beyond 4, numerous examples of graphs are obtained for which a realization is impossible that satisfy the given constraints. This tells us to be careful when using graph theoretic methods in even moderately large multi-agent systems. This work also gives us insight into the construction of various distributed algorithms which can be mapped on a decentralized multi-agent system.

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